Financial Econometrics – Two Promising Areas

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My view on financial econometrics

- Performing statistical analysis on financial data doesn't automatically make a work financial econometrics.
- Real financial econometrics should involve financial theory in a non-trivial way.
- May not need to be technically sophisticated, but definitely requires of posing the problem consistent with financial theory.
- Semantics aside, a clearer understanding of financial problems helps identify the relevant knowledge/tool sets needed for conducting effective financial econometrics research.

- Running a spline estimation to obtain a term structure of interest rates to me is not financial econometrics even though it may be an useful application.
- Neither will I consider running a vector autoregressive analysis or co-integration analysis on several financial variables financial econometrics.

The role of financial theory

- Financial theory is about linking one set of financial/economic variables to another.

- The link is established as a result of specifying how a set of variables evolve over time in conjunction with some assumptions on the market and market participants. This fact shouldn't be forgotten in performing econometric analysis.

- For example, repeatedly calibrating some form of exponential affine term structure model to the bond price data, say every day, can create an illusion of good performance.

- Such an implementation is completely at odd with the theory that gives rise to the model, because the theory relies on the assumption that the same dynamic system governs the short-term interest rates for the whole time.

The first promising area

Latent state variables are common in financial models: unobservable factor model, term structure models using instantaneous interest rates, structural or reduced-form credit risk models, CDO modeling using copulas, stochastic volatility models ...

- Viewing them as state-space econometric models, particulary non-linear and non-Gaussian type, is most appropriate.
- Many applications can be readily conducted using the existing state-space methods.
- There are many improvement possibilities on the methodological aspect of non-linear non-Gaussian state-space models.

The second promising area

Derivatives can be used to better understand the underlying dynamics: VIX, forward looking beta and risk premium, stochastic volatility specification ....

- This type requires a good understanding of derivatives pricing models.
- One will find time series econometrics to be extremely useful.

A latent variable model

Term structure models using the instantaneous interest rate

The instantaneous interest rate is assumed to obey the following dynamic:

$$dr_t = \kappa(m - r_t)dt + \sigma r_t^{\delta} dW_t$$

The Vasicek (1977) model corresponds to $\delta = 0$, and the Cox-Ingersoll-Ross (1985) model corresponds to $\delta = 0.5$.

Use an arbitrage argument and assume a constant risk premium to derive the interest rate of maturity τ (zero-coupon Treasury bond) as

$$R_t(\tau;\theta) = A(\tau;\theta) + B(\tau;\theta) r_t$$

with θ denoting the relevant parameter set.

A latent variable model (continued)

Implementation

If we use one time series of T observations, then we can write down the likelihood function for $\{r_t; t = 1, 2, \cdots\}$. But we only observe $\{R_t(\tau_t; \theta); t = 1, 2, \cdots\}$. Thus, we need to commit to some econometric formulation based on the theoretical model.

Specification 1:

$$R_t(\tau_t; \theta) = A(\tau_t; \theta) + B(\tau_t; \theta) r_t$$

The transformed-data method can be used to deal with estimation.

A latent variable model (continued)

Specification 2:

$$R_t(\tau_t;\theta) = A(\tau_t;\theta) + B(\tau_t;\theta)r_t + \varepsilon_t$$

- This case becomes a state-space model. For Vasicek (1977), i.e., $\delta = 0$, it becomes a linear Gaussian system and parameter estimation can be performed with the use of the Kalman filter.
- For the Cox-Ingersoll-Ross (1985), i.e., $\delta = 0.5$, it becomes a linear non-Gaussian system for which the Kalman filter is no longer appropriate.
- There are several approximate Kalman filters being suggested in the literature, but particle filter is a more natural solution technique.

A latent variable model (continued)

A more appropriate way of conducting estimation is to use many available interest rate time series together.

$$
\begin{aligned}
R_t(\tau_{1,t};\theta) &= A(\tau_{1,t};\theta) + B(\tau_{1,t};\theta)r_t + \epsilon_{1,t} \\
R_t(\tau_{2,t};\theta) &= A(\tau_{2,t};\theta) + B(\tau_{2,t};\theta)r_t + \epsilon_{2,t} \\
& \cdots \\
R_t(\tau_{k,t};\theta) &= A(\tau_{k,t};\theta) + B(\tau_{k,t};\theta)r_t + \epsilon_{3,t}
\end{aligned}
$$

In the multiple time series situation, one must include some ϵ's.

A latent variable model (continued)

- If one arbitrarily sets one ϵ to zero, then the transformed-data method continues to apply. Otherwise, it becomes a state-space econometric specification again.
- Many new term structure models with stochastic volatility and/or jumps should be formulated as linear non-Gaussian state-state specifications. But in the literature, they were estimated by arbitrarily removing some ϵ's.
- If one incorporates some swap rates or other interest-rate derivatives into the data set, the appropriate econometric specification becomes a non-linear non-Gaussian state-space model.

Use VIX to estimate the stochastic volatility-jump models

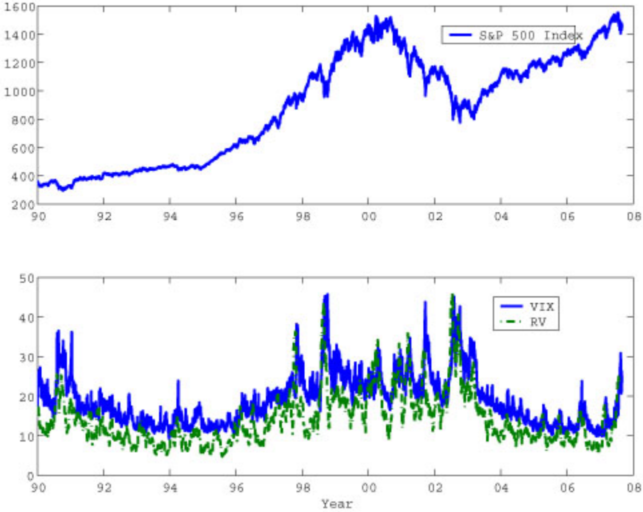
Duan and Yeh (2008) approaches the estimation of a fairly general stochastic volatility-jump using asset returns and VIX.

Their model under the physical probability measure P,

$$
\begin{aligned}
d\ln S_t &= \left[r - q + \delta_S V_t - \frac{V_t}{2}\right] dt + \sqrt{V_t} dW_t + J_t dN_t - \lambda \mu_J dt \\
dV_t &= \kappa(\theta - V_t)dt + \upsilon V_t^{\gamma} dB_t
\end{aligned}
$$

- W_t and B_t are two correlated Wiener processes with the correlation coefficient ρ.
- N_t is a Poisson process with intensity λ and independent of W_t and B_t.
- J_t is an independent normal random variable with mean μ_J and standard deviation σ_J.

The S&P 500 index, the VIX index and the corresponding realized volatility



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Use VIX to estimate the sv-jump models (continued)

The theory behind VIX

By adopting some pricing kernel, the system under the risk-neutral probability measure Q becomes

$$\begin{aligned} d\ln S_t &= \left[r - q - \frac{V_t}{2} + \lambda^* \left(\mu_J^* + 1 - e^{\mu_J^* + \frac{\sigma_J^2}{2}} \right) \right] dt + \sqrt{V_t} dW_t^* \\ &\quad + J_t^* dN_t^* - \lambda^* \mu_J^* dt \\ dV_t &= (\kappa\theta - \kappa^* V_t)\, dt + v V_t^{\gamma} dB_t^* \end{aligned}$$

where $\kappa^* = \kappa + \delta_V$ and $B_t^* = B_t + \frac{\delta_V}{v} \int_0^t V_s^{1-\gamma} ds$ with δ_V being interpreted as the volatility risk premium.

Use VIX to estimate the sv-jump models (continued)

Fact 1: The risk-neutral expected cumulative return

$$
\begin{aligned}
& E_t^Q\left(\ln \frac{S_{t+\tau}}{S_t}\right) \\
= & (r-q)\tau - \frac{1}{2}\int_t^{t+\tau} E_t^Q\left(V_s\right) ds \\
& + \int_t^{t+\tau} \lambda^* E_t^Q\left(\mu_J^* + 1 - e^{\mu_J^* + \frac{\sigma_J^2}{2}}\right) ds \\
= & \left[r - q - \lambda^*\left(e^{\mu_J^* + \frac{\sigma_J^2}{2}} - (\mu_J^* + 1)\right)\right]\tau - \frac{1}{2}\int_t^{t+\tau} E_t^Q\left(V_s\right) ds
\end{aligned}
$$

where

$$
\int_t^{t+\tau} E_t^Q\left(V_s\right) ds = \frac{\kappa\theta}{\kappa^*}\left(\tau - \frac{1 - e^{-\kappa^*\tau}}{\kappa^*}\right) + \frac{1 - e^{-\kappa^*\tau}}{\kappa^*} V_t.
$$

Use VIX to estimate the sv-jump models (continued)

Consider an option portfolio:

$$\begin{aligned} & \Pi_{t+\tau}(K_0, t+\tau) \\ \equiv\ & \int_0^{K_0} \frac{P_{t+\tau}(K; t+\tau)}{K^2} dK + \int_{K_0}^{\infty} \frac{C_{t+\tau}(K; t+\tau)}{K^2} dK \\ =\ & \frac{S_{t+\tau} - K_0}{K_0} - \ln \frac{S_t}{K_0} - \ln \frac{S_{t+\tau}}{S_t} \end{aligned}$$

Thus, taking the risk-neutral expectation gives rise to

$$e^{r\tau} \Pi_t(K_0, t+\tau) = \frac{F_t(t+\tau) - K_0}{K_0} - \ln \frac{S_t}{K_0} - E_t^Q \left(\ln \frac{S_{t+\tau}}{S_t} \right)$$

where $F_t(t+\tau)$ denotes the forward price at time t with a maturity at time $t+\tau$.

Use VIX to estimate the sv-jump models (continued)

CBOE launched the new VIX in 2003 using the following definition:

$$\mathrm{VIX}_t^2(\tau) \equiv \frac{2}{\tau} e^{r\tau} \Pi_t(F_t(t+\tau), t+\tau).$$

Using Facts 1 and 2 yields

$$\mathrm{VIX}_t^2(\tau) = 2\phi^* + \frac{1}{\tau}\int_t^{t+\tau} E_t^Q(V_s)\, ds$$

where $\phi^* = \lambda^* \left(e^{\mu_J^* + \sigma_J^2/2} - 1 - \mu_J^* \right)$.

Note: The extra term, ϕ^*, is entirely due to jumps. If the jump magnitude is small, this term is negligible.

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Use VIX to estimate the sv-jump models (continued)

Log-likelihood function

Denote the observed data series by $X_{t_i} = (\ln S_{t_i}, \mathrm{VIX}_{t_i})$. Let $\widehat{Y}_{t_i}(\Theta) = (\ln S_{t_i}, \widehat{V}_{t_i}(\Theta))$ where $\widehat{V}_{t_i}(\Theta)$ is the inverted value evaluated at parameter value Θ.

$$
\begin{aligned}
& \mathcal{L}\left(\Theta; X_{t_1}, \cdots, X_{t_N}\right) \\
= \; & \sum_{i=1}^{N} \ln f\left(\widehat{Y}_{t_i}(\Theta) \middle| \widehat{Y}_{t_{i-1}}(\Theta); \Theta\right) - N \ln\left(\frac{1 - e^{-\kappa^* \tau}}{\kappa^* \tau}\right)
\end{aligned}
$$

where

$$
f\left(\widehat{Y}_{t_i}(\Theta) \middle| \widehat{Y}_{t_{i-1}}(\Theta); \Theta\right) = \sum_{j=0}^{\infty} \frac{e^{-\lambda h_i} (\lambda h_i)^j}{j!} g\left(\mathbf{w}_{t_i}(j, \Theta); \mathbf{0}, \mathbf{\Omega}_{t_i}(j, \Theta)\right).
$$

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Use VIX to estimate the sv-jump models (continued)

$h_i = t_i - t_{i-1}$, $g(\cdot; \mathbf{0}, \mathbf{\Omega}_{t_i}(j,\Theta))$ is a bivariate normal density function with mean $\mathbf{0}$ and variance-covariance matrix:

$$\mathbf{\Omega}_{t_i}(j,\Theta) = \begin{bmatrix} \widehat{V}_{t_{i-1}}(\Theta)h_i + j\sigma_J^2 & \rho v \widehat{V}_{t_{i-1}}^{0.5+\gamma}(\Theta)h_i \\ \rho v \widehat{V}_{t_{i-1}}^{0.5+\gamma}(\Theta)h_i & v^2 \widehat{V}_{t_{i-1}}^{2\gamma}(\Theta)h_i \end{bmatrix},$$

and

$$\begin{aligned} &\mathbf{w}_{t_i}(j,\Theta) \\ = &\begin{bmatrix} \ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) - \left[r - q + (\delta_s - \frac{1}{2})\widehat{V}_{t_{i-1}}(\Theta)\right] h_i - (j - \lambda h_i)\mu_J \\ \widehat{V}_{t_i}(\Theta) - \widehat{V}_{t_{i-1}}(\Theta) - \kappa\left(\theta - \widehat{V}_{t_{i-1}}(\Theta)\right) h_i \end{bmatrix}. \end{aligned}$$

Use VIX to estimate the sv-jump models (continued)

If we believe the relationship between VIX and the latent volatility is masked by some measurement error, i.e.,

$$\mathrm{VIX}_t^2(\tau) = 2\phi^* + \frac{1}{\tau}\int_t^{t+\tau} E_t^Q\left(V_s\right) ds + \epsilon_t$$

then the appropriate specification is a linear non-Gaussian state-space model.

Particle filter can be used to solve the problem.

Concluding remarks

- As the two examples show, financial theory plays a critical role in model estimation.
- Non-linear and/or non-Gaussian state-space models emerge naturally in financial econometrics. In order to avoid such a filtering problem, one will need to make some unappealing implementation assumption.
- Combining the derivative contracts with the underlying asset opens many possibilities in financial econometrics research both in terms of application and methodological development.