From Models to Data in Macroeconomic Research

an Overview

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Phillips Curve

Phillips (1958, Economica) proposes: the rate of change in money wage rates can be explained by the level of unemployment.

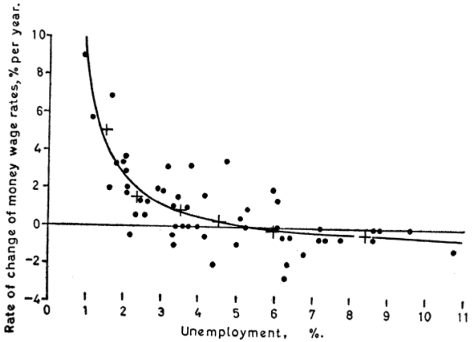


Fig.1. 1861 – 1913

Keynesian vs. Friedman's View

- Keynesian: high inflation is a price to pay to get low unemployeemet (or economic growth).
- Friedman(1968, AER) : "there is always a temporary trade-off between inflation and unemployment, there is no permanent trade-off. The temporary trade-off comes...from unanticipated inflation" (p.11).
    - Policy makers cannot continue to fool rational agents by systematically exploiting the tradeoff between unemployment and inflation.

Rational Expectations & Lucas Critique

- Friedman and others paved the way for rational expectations revolution.
- The devastating critique by Lucas (1976) has a profound influence on macroeconometrics:
    - "...that optimal decision rules vary systematically with changes in the structure of series relevant to the decision maker, it follows that any change in policy will systematically alter the structure of econometric models" (p. 41).

Reduced Form vs. Structural Params.

- RF parameters are often functions of policy and "deep" parameters; can be unstable as a result of policy changes
- It is dangerous to rely on RF estimation to predict the effects under an alternative policy.
- The stagflation in the 1970s served as a example of unstable RF relationship between inflation and unemployment.

Cagan Money Demand (Cagan, 1956)

Consider a money demand[1]

$$\log \frac{M_t^d}{P_t} = -\beta E_t \log \frac{P_{t+1}}{P_t} + \varepsilon_t^d,\ \beta > 0$$

Let small letters denote variables in logs. Then, money demand is

$$m_t^d - p_t = -\beta \left(E_t p_{t+1} - p_t \right) + \varepsilon_t^d$$

Suppose money supply follows an exogenous process,

$$m_t^s = \rho m_{t-1}^s + \varepsilon_t^s,\ |\rho| < 1$$

[1] This example is taken from lecture notes by Eric Leeper.

Structural Form of the Equilibrium

- The equilibrium market clearing condition is

$$m_t^s = m_t^d \equiv m_t.$$

- The model is a linear rational expectations model. The solution is

$$\begin{bmatrix} p_t \\ m_t \end{bmatrix} = \begin{bmatrix} 0 & \frac{\rho}{1+\beta(1-\rho)} \\ 0 & \rho \end{bmatrix} \begin{bmatrix} p_{t-1} \\ m_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & \frac{\rho}{1+\beta(1-\rho)} \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \varepsilon_t^d \\ \varepsilon_t^s \end{bmatrix} \tag{1}$$

- From the solution, we can derive impulse responses to shocks with economic interpretations, ε_t^d & ε_t^s.

Reduced Form of the Equilibrium

- An econometrician with data on price and money balance can easily estimate a RF VAR.

$$\begin{bmatrix} p_t \\ m_t \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} p_{t-1} \\ m_{t-1} \end{bmatrix} + \begin{bmatrix} u_t^p \\ u_t^m \end{bmatrix}, \qquad (2)$$

  where u_t^p and u_t^m are one-step ahead forecast errors.
- Comparing (2) with (1), we have $b_{11} = b_{21} = 0$, $b_{12} = \frac{\rho}{1+\beta(1-\rho)}$, $b_{22} = \rho$, and a mapping between one-step ahead forecast errors ($u's$) and structural errors ($\varepsilon's$):

$$u_t^p = \frac{1}{1+\beta(1-\rho)}\varepsilon_t^d + \frac{1}{\beta+1}\varepsilon_t^s; \; u_t^m = \varepsilon_t^s.$$

The Business of Empirical Macro

- After Lucas Critique, empirical macro is about the identification of structural parameters or structural shocks from RF estimates.
- Impulse responses based on RF errors (one-step ahead forecast errors) do not help us understand economic behaviors and cannot allow us to answer counterfactual policy questions.

RBC Literature and DSGE Models

Rebelo (2005) summarizes Kydland and Prescott's three revolutionary ideas, which together with Lucas and others, build the prototype of the analytical framework for modern macro:

- Studying BC in a dynamic stochastic general equilibrium (DSGE) model.
  - individual and firms' behaviors are modeled with micro foundation.
  - agents form rational expectations about the future (i.e. they know the structure of the economy and cannot make systematic errors in their predictions)

RBC Literature and DSGE Models

- Insisting that BC models must be consistent with the empirical regularities of long-run growth, qualitatively and quantitatively.
- Calibrating models with parameters drawn from microeconomic estimation and long-run properties of the economy $\Rightarrow$ generating data to be compared with actual data for important BC statistics.

A Simple RBC Model (Cooley, 1995)

Households choose C and L to maximize utility

$$Max\ E_t \sum_{t=0}^{\infty} \beta^t (1+\eta)^t \left[(1-\alpha)\log c_t + \alpha \log(1-h_t)\right], \text{ s.t.}$$

$$c_t + x_t = r_t k_{t-1} + w_t h_t$$

$$(1+\gamma)(1+\eta) k_t = (1-\delta) k_{t-1} + x_t.$$

Firms solve a profit maximization problem

$$Max\ e^{z_t} (1+\gamma)^{t(1-\theta)} k_{t-1}^{\theta} h_t^{1-\theta} - r_t K_{t-1} - w_t H_t.$$

where z_t is a random productivity parameter, the source of uncertainty in the economy, evolving according to

$$z_t = \rho z_{t-1} + \varepsilon_t,\ \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

Calibration

Need to consider the correspondence between the model economy and the measurements that are take to produce data.

Technology					Preferences			
θ	δ	ρ	σ_ε	γ	β	σ	α	η
.4	.012	.95	.007	.0156	.987	1	.64	.012

Calibration

Given the simple structure, the model produces BC characteristics, comparable reasonable well to some key aspects of BC statistics.

	relative s.d.%		corr. w/ output	
variables	model	data	model	data
Output	1	1	1	1
Consumption	.24	.5	.84	.83
Investment	4.4	4.8	.99	.91
Hours	.57	.92	.99	.86
Productivity	.45	.43	.98	.66

Data are based on U.S. 1964:I-1991:II

From Data to Models

Calibration is crude and informal, but it's a beginning step to connect theory and data. The modern macro literature goes in two directions:

- *Empirical findings help improve theory*:
  - mismatch of correlation of certain variables: adding new features to improve RBC models' predictibility of observed empirical regularity. e.g., high correlations b/w hours and productivity implied in a simple RBC–adding distorting tax disturbances
  - improving the qualitative response patterns–adding habit formation and investment/capital adjustment costs to get hump-shaped responses

From Models to Data

- *Theory provides restrictions to help identification in data*:
  - Less restrictive approach: structural VAR
  - More restrictive approach: increasingly complex models are taken into data, such as estimation of DSGE models

The rest of the talk will focus on structural VAR and DSGE estimation by ML.

Procedure

- Selecting a system of observables ($\mathbf{X}_t$), which characterize the eqm of interest (nontrivial)
- Estimating RF by OLS:

$$\mathbf{X}_t = \mathbf{B}(\mathbf{L})\mathbf{X}_{t-1} + \mathbf{u}_t, \quad E(\mathbf{u}_t \mathbf{u}_t') = \Omega$$

- The underlying SF is

$$\begin{aligned} \mathbf{A}_0 \mathbf{X}_t &= \mathbf{A}(\mathbf{L})\mathbf{X}_{t-1} + \varepsilon_t \\ \mathbf{E}(\varepsilon_t \varepsilon_\tau') &= \begin{cases} \mathbf{D}, \text{ for } t = \tau \\ \mathbf{0}, \text{ otherwise} \end{cases} \end{aligned}$$

$\mathbf{D}$ is diagonal.

Procedure

- Comparing SF to RF, we have

$$\mathbf{A}(\mathbf{L}) = \mathbf{A}_0 \widehat{\mathbf{B}}(\mathbf{L})$$

$$\varepsilon_t = \mathbf{A}_0 \widehat{\mathbf{u}}_t$$

$$\widehat{\Omega} = E\left[\mathbf{A}_0^{-1} \varepsilon_t \varepsilon_t' \left(\mathbf{A}_0^{-1}\right)'\right] = \mathbf{A}_0^{-1} \mathbf{D} \left(\mathbf{A}_0^{-1}\right)'$$

- Identification of the structural model follows from imposing sufficient restrictions on $\mathbf{A}_0$ so that there are no more than $\frac{n(n-1)}{2}$ free parameters in $\mathbf{A}_0$.

Estimation: $\mathbf{A}_0$ & $\mathbf{D}$

Choosing $\mathbf{A}_0$ and $\mathbf{D}$ to maximize

$$L\left(\mathbf{A}_0, \mathbf{D}, \widehat{\mathbf{B}}(\mathbf{L})\right) = -\left(\frac{Tn}{2}\log 2\pi\right) - \frac{T}{2}\log\left|\mathbf{A}_0^{-1}\mathbf{D}\left(\mathbf{A}_0^{-1}\right)'\right| - \frac{1}{2}\sum_{t=1}^{T}\widehat{\mathbf{u}}_t'\left[\mathbf{A}_0^{-1}\mathbf{D}\left(\mathbf{A}_0^{-1}\right)'\right]^{-1}\widehat{\mathbf{u}}_t$$

or

$$L\left(\mathbf{A}_0, \mathbf{D}, \widehat{\mathbf{B}}(\mathbf{L})\right) = -\left(\frac{Tn}{2}\log 2\pi\right) + \frac{T}{2}\log\left|\mathbf{A}_0\right|^2 - \frac{T}{2}\log\left|\mathbf{D}\right| - \frac{T}{2}tr\left\{\left(\mathbf{A}_0'\mathbf{D}^{-1}\mathbf{A}_0\right)\widehat{\Omega}\right\}$$

Impulse Responses

- After obtaining $\mathbf{A}_0$, derive the MA representation for the SF to get IR for $\varepsilon' s$.

$$\mathbf{X}_t = [\mathbf{A}_0 - \mathbf{A}(\mathbf{L})\mathbf{L}]^{-1} \varepsilon_t \equiv \mathbf{C}(\mathbf{L})\varepsilon_t$$

- Unfinished business about identification: checking IR to the shock(s) of interest. Two scenarios–
  - If IR consistent with theory, declare identification achieved.
  - If IR inconsistent with theory, try a new set of identifying restrictions. (Alternatively, if robust empirical evidence is found, go back to fix theory.)

An Example of 3-Var System

- $\mathbf{X}_t = [G_t, T_t, Y_t]$, Cholesky decomposition

$$\mathbf{A}_0' = \begin{array}{cccc} & G & T & Y \\ G & 1 & X & X \\ T & 0 & 1 & X \\ Y & 0 & 0 & 1 \end{array}$$

- Blanchard and Perotti (2002)

$$\mathbf{A}_0' = \begin{array}{cccc} & G & T & Y \\ G & 1 & X & X \\ T & 0 & 1 & X \\ Y & 0 & -2 & 1 \end{array}$$

Distribution of Impulses Responses (1)

To assess the statistical significance of the dynamics induced by certain shocks, we need standard errors. Three methods:

- δ method
    - Asymptotic distribution
    - IRs are differentiable functions of the VAR parameters and of the covariance matrix.
    - Rarely used, estimated VAR coefficients have large asymptotic standard errors, often concluding insignificant responses at all horizons. The cost of the generality of VAR, which imposes few restrictions on the dynamics is that the inferences drawn are not too precise.

Distribution of Impulses Responses (2)

- Bootstrap
  1. Obtain $\widehat{\mathbf{B}}(\mathbf{L})$ and $\widehat{\mathbf{u}}_t$ by OLS.
  2. Obtain $\mathbf{u}_t^i$ by sampling from $\{\widehat{\mathbf{u}}_t\}$ with replacement. Construct $\mathbf{X}_t^i = \widehat{\mathbf{B}}(\mathbf{L})\mathbf{X}_{t-1}^i + \mathbf{u}_t^i$, $i = 1, ..., I$.
  3. Estimate $\widehat{\mathbf{B}}(\mathbf{L})^i$ using data constructed in step 2, and imposing the same identifying restrictions to get $\widehat{\mathbf{C}}(\mathbf{L})^i$.
  4. Report percentiles of the distribution of $\mathbf{C}(\mathbf{L})$.

Distribution of Impulses Responses (3)

- Monte Carlo (See p. 136 in Canova (2007))
  - Sampling from the assumed distribution of the white noise process.
  - Under certain assumptions, the likelihood of VAR estimates is the product of normal density for RF coefficient and a Wishart density for the residual inverse.
  - Generate a large number of estimates for RF coefficients and var-cov matrix.
  - Apply identifying restrictions and compute impulse responses.
  - Report percentiles of the distributions.

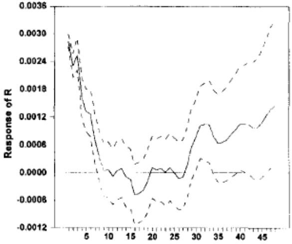
MP in a Small Open Economy

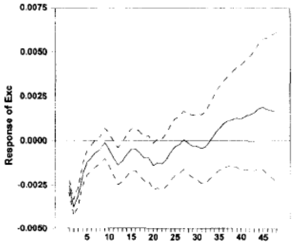
Cushman and Zha (JME, 1997)

- When taking home interest rate innovations to represent monetary policy shocks, identifying restrictions under Cholesky approach produce “price puzzle” and “exchange rate puzzle.”
- Instead, estimating a structural VAR, accounting features of small open economy, resolves these puzzles.

IR under Cholesky Factorization

- Variable order:
  $\mathbf{X}_t = [Y^*, P^*, R^*, Wxp^*, Tm, Tx, Y, P, R, M1, Exc]$
- Under Cholesky decomposition, a contractionary MP shock depreciates Canadian currency





IR under C&Z's Identification

	Exc	M1	R	Tm	Tx	Y	P	Y*	P*	R*	Wxp*
Exc	1		X					0	0	0	0
M1	X	1	X					0	0	0	0
R	X	X	1					0	0	0	0
Tm	X	0	0	1				0	0	0	0
Tx	X	0	0		1			0	0	0	0
Y	X	X	0			1		0	0	0	0
P	X	X	0				1	0	0	0	0
Y*	X	0	0					1			
P*	X	0	0						1		
R*	X	0	X							1	
Wxp*	X	0	X								1

Estimated Contemporaneous Coeff.

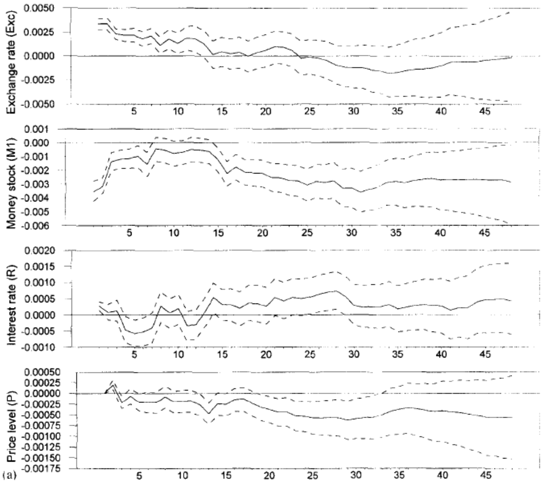
The estimated money demand

$$\mathbf{21.06}(M1 - P) - \mathbf{21.06}y + \mathbf{2.75}R = \varepsilon^d$$

The esitmated money supply

$$\mathbf{1.53}R - \mathbf{113.55}M1 + \mathbf{163.63}Exc + 0.16R^* - \mathbf{10.87}Wxp^* = \varepsilon^s$$

IR under Alternative Identification



Basic Ideas

- Recall $E(\mathbf{u}_t \mathbf{u}_t') = \Omega \ = \mathbf{A}_0^{-1} \left(\mathbf{A}_0^{-1} \right)'$
- Let $\mathbf{A}_0^{-1} \equiv T$ be the lower triangular matrix from Cholesky factorization of Ω, which contains impact responses.
- Consider an orthonormal matrix, Q. Then, we have

$$\widehat{\mathbf{u}}_t = T\varepsilon_t = TQ'Q\varepsilon_t \equiv T^* \varepsilon_t^*,$$

where T^* is another impact responses corresponding to another set of identified shocks ε_t^*. The two structural models, one associated with $T\varepsilon_t$ and the other with $T^*\varepsilon_t^*$, have the same RF.

Basic Ideas

- The key idea is to generate many $Q's$ and toss off those producing responses inconsistent with the sign restrictions imposed.
- Where do the restrictions come from? Again, from structural models.
- Uhlig calls this an agnostic identification scheme. Why? While this approach can impose strong prior on the qualitative responses. In practice, we impose restrictions that are least controversial.

Generating $Q's$ Systematically

- Householder transformations
  - Draw a matrix $W \sim N(0, I_n)$
  - Apply QR decomposition to $W = QR$, where Q is orthonormal and R is triangular
- Givens transformations:
  - Determine the format. In a 3-var system, a Givens matrix can take any of the three forms

$$Q_{12} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, Q_{13} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$Q_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

Generating $Q's$ Systematically

For each above Q, $QQ' = I$. In practice, can set

$$Q_G = Q_{12}(\theta_1) \times Q_{13}(\theta_2) \times Q_{13}(\theta_3)$$

where each θ is randomly selected from uniform $[0, \pi]$ or $[0, \frac{1}{2}\pi]$. Q_G is still an orthonormal matrix.

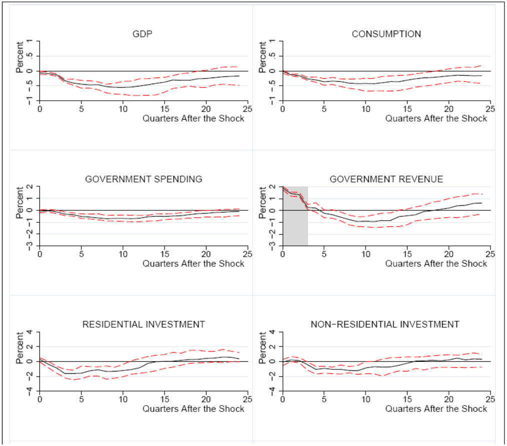
An Example: Mountford-Uhlig (2006)

	Gov. Revenue	Gov Spending	GDP, Cons, Non-Res Inv	Interest Rate	Adjusted Reserves	Prices
Other Shocks						
Business Cycle	+		+			
Monetary Policy				+	-	-
Basic Fiscal Policy Shocks						
Government Revenue	+					
Government Spending		+				

A Word of Caution

- Paustian (2007) shows that a sufficiently large number of restrictions must be imposed to deliver unambiguously the correct sign of the unconstrained impulse response.

IR to a Govt Spending Shock



From SVAR to DSGE

- Seem remote but close: Later we will show the eqm of DSGE models can be seen as a structural VAR. Coefficients are (complicated) functions of structural parameters, which describe preference, policy, technology, etc.
- Most SVAR's impose a small set of restrictions implied by DSGE models to obtain identification.
- DSGE estimation, on the other extreme, impose all the cross-equation restrictions implied by the models.

Basic Ideas of Bayesian Approach

1. Specify a rich DSGE model.
2. Find the steady state, log-linearize eqm conditions, and solve for the linear rational expectations eqm.
3. Characterize the eqm by a state-space form.
4. Draw parameters from assumed prior distributions.
5. Use the Kalman filter to get predictions for the “unobserved” state variables.
6. Compute the likelihood for the observables.
7. After many draws, report the distribution of parameters.

Setup: Households

A DSGE model to evaluate U.S. monetary policy.

- A representative household supplies $H_t(i)$ units of labor at nominal wage W_t and $K_t(i)$ units of capital at the nominal rental rate R_t to each intermediate goods-producing firm $i \in [0, 1]$. Total labor and capital supplied are

$$H_t = \int_0^1 H_t(i)di,$$

$$K_t = \int_0^1 K_t(i)di$$

Setup: Households

Households solve the utility maximization problem

$$E_t \sum_{i=1}^{\infty} \beta^t \left\{ \frac{\gamma}{1-\gamma} \ln \left[C_t^{\frac{\gamma}{1-\gamma}} + b_t \left(\frac{M_t}{P_t} \right)^{\frac{\gamma}{1-\gamma}} \right] + \eta \ln (1 - H_t) \right\},$$

s.t. the budget constraint

$$C_t + K_t - (1-\delta) K_{t-1} + \frac{M_t}{P_t} = \frac{M_{t-1} + T_t + W_t H_t + R_t K_{t-1} + D_t}{P_t}$$

b_t is money demand disturbance, following

$$\ln b_t = \rho_b \ln(b_{t-1}) + \varepsilon_t^b, \;\; \rho_b \in (-1, 1), \varepsilon_t^b \sim N(0, \sigma_b^2)$$

Setup: Interm. Goods-Producing Firms

The production function of intermediate firm i takes the form of

$$Y_t(i) = A_t K_i(t)^{\alpha} \left[g^t H_t(i)\right]^{1-\alpha}$$

where A_t follows an AR process

$$\ln A_t = (1 - \rho_A) \ln A + \rho_A \ln A_{t-1} + \varepsilon_t^A,$$

where $\rho_A \in (-1, 1)$ and $\varepsilon_t^A \sim N(0, \sigma_A^2)$.

Setup: Interm. Goods-Producing Firms

To allow monetary policy have real effects, we assume each intermediate goods-producing firm has price-setting power and faces a cost of adjusting nominal price, give by

$$\frac{\phi}{2}\left[\frac{P_t(i)}{P_{t-1}(i)}-1\right]^2 Y_t.$$

Intermediate goods-producing firms choose $H_t(i)$, $K_t(i)$, $Y_t(i)$, and $P_t(i)$ to maximize the total market value of the firm

$$E_t\sum_{i=1}^{\infty}\beta^t\frac{\Lambda_t}{P_t}\left\{\begin{array}{c} P_t(i)Y_t(i)-W_tH_t(i)-R_tK_{t-1}(i) \\ -P_t\frac{\phi}{2}\left[\frac{P_t(i)}{P_{t-1}(i)}-1\right]^2 Y_t \end{array}\right\}$$

Setup: Finished Goods-Producing Firms

The finished goods-producing firm uses $Y_t(i)$ units of each intermediate good i to produce finished goods Y_t

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{\theta}{\theta-1}} di \right]^{\frac{\theta}{\theta-1}},$$

A finished goods-producing firm chooses $Y_t(i)$ and Y_t to maximize its profit, implying the demand for $Y_t(i)$ is

$$Y_t(i) = \left[\frac{P_t(i)}{P_t} \right]^{-\theta} Y_t$$

where $-\theta$ is the final goods-producing firms' demand elasticity for good i.

Setup: Monetary Policy

Define the gross money growth rate as $\mu_t \equiv \frac{M_t}{M_{t-1}}$. The central bank adopts a money supply rule

$$\ln \mu_t = \rho_\mu \ln \mu_{t-1} + \psi_A \varepsilon_t^A + \psi_b \varepsilon_t^b + \varepsilon_t^\mu .$$

Government budget constraint is

$$M_t = M_{t-1} + T_t .$$

Seigniorage revenues are used to make lump-sum transfers to households.

Solving DSGE: Eqm conditions

- Equilibrium conditions: FOC conditions, market clearing conditions, policy rules, stochastic process for shocks, etc., most highly non-linear, e.g. FOC for C_t and $\frac{M_t}{P_t}$

$$\frac{C_t^{-\frac{1}{\gamma}}}{C_t^{\frac{\gamma-1}{\gamma}} + b_t\left(\frac{M_t}{P_t}\right)^{\frac{\gamma-1}{\gamma}}} = \Lambda_t$$

$$\frac{b_t\left(\frac{M_t}{P_t}\right)^{-\frac{1}{\gamma}}}{C_t^{\frac{\gamma-1}{\gamma}} + b_t\left(\frac{M_t}{P_t}\right)^{\frac{\gamma-1}{\gamma}}} = \Lambda_t - \beta E_t\left(\frac{\Lambda_{t+1}P_t}{P_{t+1}}\right)$$

Solving DSGE: Steady State

- The ss has the interpretation of long-run performance of the economy.
- The ss demonstrates balanced growth path: detrended output, consumption, and capital are constants.
- The ss eqm can be characterized by a system of 10 equations with 10 unknowns: given a set of values of parameters, the solution of ss gives the all endogenous variables in the ss.

Solving DSGE: Log-linearization

Examples of log-linearized eqm conditions:

- FOC for c_t

$$-\frac{1}{\gamma}c^{-\frac{1}{\gamma}}\widehat{c}_t = \lambda c^{\frac{\gamma-1}{\gamma}}\left[\widehat{\lambda}_t + \left(\frac{\gamma-1}{\gamma}\right)\widehat{c}_t\right] + \lambda b m^{\frac{\gamma-1}{\gamma}}\left[\widehat{\lambda}_t + \widehat{b}_t + \left(\frac{\gamma-1}{\gamma}\right)\widehat{m}_t\right]$$

- FOC for $\frac{M_t}{P_t}$

$$\left[bm^{-\frac{1}{\gamma}} - \lambda bm^{\frac{\gamma-1}{\gamma}} + \frac{\beta\lambda}{g\pi}bm^{\frac{\gamma-1}{\gamma}}\right]\widehat{b}_t -$$

$$[\frac{bm^{-\frac{1}{\gamma}}}{\gamma} + \lambda\left(\frac{\gamma-1}{\gamma}\right)bm^{\frac{\gamma-1}{\gamma}} - \frac{\beta\lambda}{g\pi}\left(\frac{\gamma-1}{\gamma}\right)bm^{\frac{\gamma-1}{\gamma}}]\widehat{m}_t -$$

$$\left[\lambda c^{\frac{\gamma-1}{\gamma}} + \lambda bm^{\frac{\gamma-1}{\gamma}}\right]\widehat{\lambda}_t - \left[\lambda c^{\frac{\gamma-1}{\gamma}}\left(\frac{\gamma-1}{\gamma}\right) - \frac{\beta\lambda}{g\pi}c^{\frac{\gamma-1}{\gamma}}\left(\frac{\gamma-1}{\gamma}\right)\right]\widehat{c}_t +$$

$$\left[\frac{\beta\lambda}{g\pi}\left(c^{\frac{\gamma-1}{\gamma}} + m^{\frac{\gamma-1}{\gamma}}\right)\right]\left(\widehat{\lambda}_{t+1} - \widehat{\pi}_{t+1}\right) - \eta_t = 0$$

Linear Rational Expectations Solution

- After log-linearizing all the eqm conditions, the dynamics of the economy can represented by the system

$$\mathbf{\Gamma}_0\mathbf{X}_t = \mathbf{\Gamma}_1\mathbf{X}_{t-1} + \mathbf{\Psi}\varepsilon_t + \mathbf{\Pi}\eta_t$$

where $\mathbf{X}_t = [\widehat{y}_t, \widehat{H}_t, \widehat{w}_t, \widehat{r}_t, \widehat{c}_t, ...]$; $\varepsilon_t = [\varepsilon_t^A, \varepsilon_t^b, \varepsilon_t^\mu]$; η_t is the vector of one step ahead forecast errors.

- Applying Sims's (2002) algorithm, the solved model has the following representation

$$\mathbf{X}_t = \mathbf{G}\mathbf{X}_{\mathbf{t-1}} + \mathbf{M}\varepsilon_t, \text{ (Look familiar?)}$$

where elements in $\mathbf{G}$ and $\mathbf{M}$ are functions of parameters and ss values of endogenous variables.

State-Space Representation

The solution, $\mathbf{X}_t = \mathbf{G}\mathbf{X}_{\mathbf{t-1}} + \mathbf{M}\varepsilon_t$, is rearranged into a state-space form, which consists of state equations:

$$\xi_{t+1} = \mathbf{F}\xi_t + \mathbf{v}_{t+1}$$

and observation (measurement) equations:

$$\mathbf{Y}_{t+1} = \mathbf{H}'\xi_t + \mathbf{w}_{t+1}$$

- In general, set $\mathbf{F}\xi_t = \mathbf{X}_t$; choose those endogenous variables with data or better measurements as $\mathbf{Y}_t$. In this case, y, m, π.
- no. of observable equation = no. of structural shocks. In this case, ε_t^A, ε_t^b, ε_t^μ.

Prior Distributions

- Choosing appropriate functional forms or setting criteria to exclude unreasonable parameters e.g.
  - $-\gamma$ is the interest elasticity of money demand $\Rightarrow$ Theory says $\gamma > 0$
  - g is the long-run real growth rate of the economy per capita. In the U.S., the quarterly model should be around 1.005. Set prior center at this value with small s.d.
- There are parameters notoriously difficult to estimate, like β, δ if no data on capital. What to do? Calibrate them (At the extreme, if we calibrate everything, we are doing calibration rather than estimation).

Metropolis-Hastings Algorithm

Parameters to estimate θ:

$$\gamma, \rho_b, \sigma_b, g, A, \rho_A, \sigma_A, \phi, \mu, \psi_A, \psi_b, \rho_\mu, \sigma_\mu$$

- Choose the initial condition, θ_0.
- Make a candidate draw for θ by the following rule

$$\theta^{new} = \theta^{old} + z_t$$

  where $z_t \sim N(0, \lambda)$ and λ is a tuning parameter.
- Using the sample likelihood ($p(\mathbf{Y}_t|\theta)$) and prior ($p(\theta)$), calculate the likelihood of this draw.

Metropolis-Hastings Algorithm

- Compute the likelihood of the new to the old draw:

$$\gamma = \frac{\pi(\theta^{new})}{\pi(\theta^{old})}$$

- Draw $u \sim U(0,1)$. If $u \leq \gamma$, accept the draw. If not, set $\theta^{new} = \theta^{old}$.
  - If $\pi(\theta^{new}) \geq \pi(\theta^{old})$, the rule accepts the new.
  - If $\pi(\theta^{new}) < \pi(\theta^{old})$, the rule allows possibility to accept the new: ensuring low density areas of the distribution are visited.
- Do this for a large number to get the posterior distribution.

A Note on Kalman Filter

DSGE models contain unmeasurable variables. Use the Kalman filter to sequentially update a linear projection of the unobserved state. (See Hamilton, 1994, Ch.13)

- Some intuition
  - Begin with some initial guess of the state, ξ_1.
  - Compute the forecasted observables based on this guess.
  - Update the inference of ξ_1 based on forecast error of the observables.
  - Use the state system to compute the $\widehat{\xi}_{2|1} = \mathbf{F}\widehat{\xi}_{1|1}$
  - Repeat this process sequentially to generate linear projection of ξ_t, consistent with the DSGE structure.

Uses of DSGE Estimation

In addition to the standard exercises in calibrated models,

- Ask counterfactual questions, less susceptible to Lucas Critique
- Forecast with a structural foundations
- Historical variance decomposition
- Testing competing theories with data
- ...

Concluding Remarks

- Other issues important for linking theory with data.
  - For example, the inveritibility issues (Fernandez-Villaverde, et al, 2007, AER)
- Macro theories or models without being confronted with data are less persuasive.
- Empirical studies without digging into its underlying economic structure is less useful and may produce misleading results.