An Introduction to Copula Analysis

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Copula

Copula is one of the most hot keywords in the recent empirical finance literature. We can find countless working papers from Google Scholar that apply copula to market relationship, financial contagion, risk management, asset allocation, and many other financial problems.

In this talk, we will introduce the following concepts:

1. What is copula?
2. Why do we need copula?
3. How to establish representative copulae?
4. How to define the cross-dependence measures of a copula?

We will also deal with the econometric issues:

1. Specification: A generalized copula-based multivariate dynamic model.
2. Estimation: A simple three-stage estimation method.
3. Testing: A flexible class of moment-based tests for copula.

Reference: Chen, Y.-T. (2007). Moment-based Copula Tests for Financial Returns, *Journal of Business and Economic Statistics*, **25**, 377-397.

What is copula?

Roughly speaking, copula is a distribution of distributions. Specifically, copula is a multivariate distribution of a finite set of univariate distributions.

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Let $\mathbf{y}_t := (y_{1t}, y_{2t}, \ldots, y_{nt})^\top$ be an $n \times 1$ vector of continuous random variables at time t. Suppose that

- $F_{\mathbf{y}}$: the multivariate distribution of $\mathbf{y}_t$,
- F_i: the univariate distribution of y_{it},
- F_i^{-1}: the quantile function of y_{it}.

Define the probability integral transformation (PIT):

$$u_{it} := F_i(y_{it}) \quad \text{and} \quad u_i := F_i(y_i), \quad i = 1, 2, \ldots, n.$$

By fixing $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, note that the events:

$$A = \{y_{1t} \le y_1,\ y_{2t} \le y_2,\ \ldots,\ y_{nt} \le y_n\}$$

and

$$A' = \{u_{1t} \le u_1,\ u_{2t} \le u_2,\ \ldots,\ u_{nt} \le u_n\}$$

are identical in the sense that $\mathbb{P}(A) = \mathbb{P}(A')$.

By definition,

$$F_{\mathbf{y}}(y_1, y_2, \ldots, y_n) := \mathbb{P}(A).$$

Correspondingly, we may define a function $C : [0,1]^n \longrightarrow [0,1]$:

$$C(u_1, u_2, \ldots, u_n) := \mathbb{P}(A'),$$

and refer to it as the copula function (of $\mathbf{y}_t$). From $\mathbb{P}(A) = \mathbb{P}(A')$, it is clear that

$$F_{\mathbf{y}}(y_1, y_2, \ldots, y_n) = C(u_1, u_2, \ldots, u_n). \tag{1}$$

Sklar's theorem

In view of $y_i = F_i^{-1}(u_i)$, we can rewrite (1) as

$$C(u_1, u_2, \ldots, u_n) = F_{\mathbf{y}}(F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_n^{-1}(u_n)). \quad (2)$$

In view of $u_i = F_i(y_i)$, we can also re-express (1) as

$$F_{\mathbf{y}}(y_1, y_2, \ldots, y_n) = C(F_1(y_1), F_2(y_2), \ldots, F_n(y_n)). \quad (3)$$

The results shown in (2) and (3) are known as Sklar's (1959) theorem.

Sklar's theorem is likely the most basic and important thing in the whole copula literature.

1. It means that, for any $F_{\mathbf{y}}$, there exists a corresponding copula C.
2. It also illustrates that, by fixing $F_{\mathbf{y}}$, we can define a corresponding copula function in accordance with (2). Therefore, by choosing different $F_{\mathbf{y}}$'s, we can derive different copula functions, such as the independent, normal, Gumbel, and t copulae that we will discuss later.
3. The result in (3) indicates that the copula function is the joint distribution of the PITs. Therefore, the copula function is also known as the "dependence function."

Copula=Cross-Dependence

The copula function contains all the information regarding the cross-dependence structures of $y_{1t}, y_{2t}, \ldots, y_{nt}$. In case that we are interested in studying the cross-dependence structures of economic variables, it is therefore natural and important to investigate the underlying copula function.

Why do we need copula?

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Flexibility (or said generality) may be an important reason.
There are several ways to see the flexibility of copula in empirical finance.

Copula is flexible to generate new multivariate distributions.

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- To see this point, we consider the case where $F_{\mathbf{y}}$ is a multivariate normal distribution. This $F_{\mathbf{y}}$ implies that the associated F_{y_i}'s must also be normal. This implication is obviously too restrictive for financial data because financial returns are typically of heavy-tails.
- To take into account the heavy-tails of returns, we may replace the multivariate normal distribution with some new copula-based multivariate distributions:

  normal copula + certain heavily-tailed marginal distributions,

  as we will see in the second part of this talk.

Copula is flexible to unify existing multivariate distributions.

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- multivariate normal distribution = normal copula + normal (marginal) distributions
- multivariate t distribution = t copula + t distributions
- Gumbel's type-B bivariate distribution = Gumbel copula + extreme value distributions

Copula is flexible to explore various cross-dependence structures.

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- Conventionally, researchers are accustomed to measuring the cross-dependence of two variables by using Pearson's correlation coefficient.
- Indeed, it is the copula, rather than the correlation coefficient, that plays the role of the dependence function.
- The correlation coefficient is simply an incomplete measure on the cross-dependence structures.By using the copula, we can define other important complements, and even alternatives, to the correlation coefficient.
- Importantly, to characterize the cross-dependence structures in a complete way, what we need to known is the true copula (or the true multivariate distribution).

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Copula is also flexible for parameter estimation.

- In the case where the parameters of the marginal distributions are separable, we can estimate the parameters of the copula-based multivariate distribution in a multi-stage way.
- Specifically, we may first estimate the parameters of the marginal distributions, and then estimate the copula parameters.
- This is particularly important when the multivariate distribution is complicated and is difficult to be estimated.
- However, the asymptotic variance-covariance matrix of the multi-stage estimators is not the same as that of the one-stage estimators. The former are typically much more complicated than the latter. This is an important issue often ignored by practitioners.

Representative Copulae

To do the copula analysis in a parametric way, we have to know the closed forms and the implied cross-dependence structures of certain representative copulae. In the following, we introduce some important bivariate copulae. These copulae are all derived from (2).

Independent copula

If $F_{\mathbf{y}}$ is a distribution of two independent random variables:

$$F_{\mathbf{y}}(y_1, y_2) = F_1(y_1)F_2(y_2),$$

then (2) generates the independent copula:

$$C_I(u_1, u_2) := u_1 u_2.$$

Normal copula

If $F_{\mathbf{y}}$ is a bivariate normal distribution with the correlation coefficient $\rho \in (-1, 1)$, then (2) generates the normal copula:

$$C_N(u_1, u_2; \rho)$$
$$= \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{v_1^2 - 2\rho v_1 v_2 + v_2^2}{2(1-\rho^2)}\right) \mathrm{d}v_2 \mathrm{d}v_1,$$

where Φ^{-1} is the quantile function of $N(0, 1)$.

The t copula

If $F_{\mathbf{y}}$ is the bivariate t distribution with the parameter $\rho \in (-1, 1)$ and the degrees of freedom ν, then (2) degenerates to the t copula:

$$C_t(u_1, u_2; \rho, \nu)$$
$$= \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right) \pi\nu\sqrt{1-\rho^2}} \left(1 + \frac{v_1^2 - 2\rho v_1 v_2 + v_2^2}{\nu(1-\rho^2)}\right)^{-\left(1+\frac{\nu}{2}\right)}$$
$$\mathrm{d}v_2 \mathrm{d}v_1,$$

where t_ν^{-1} is the univariate Student's t quantile function with the degrees of freedom ν, and ρ is the correlation coefficient if $\nu > 2$.

The t copula degenerates to a normal copula as $\nu \rightarrow \infty$.

The Gumbel copula

If $F_{\mathbf{y}}$ is Gumbel's type-B bivariate extreme value distribution with the parameter $\vartheta \in (0, 1]$, then (2) becomes the Gumbel copula:

$$C_G(u_1, u_2; \vartheta) = \exp\left[-\left((-\ln u_1)^{\frac{1}{\vartheta}} + (-\ln u_2)^{\frac{1}{\vartheta}}\right)^{\vartheta}\right].$$

The Gumbel-survival copula

Given a copula C, we can define the corresponding survival copula:

$$C^s(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2).$$

By setting $C = C_G$, the resulting C^s becomes the Gumbel-survival copula:

$$C_G^s(u_1, u_2; \vartheta_s) = u_1 + u_2 - 1 + C_G(1 - u_1, 1 - u_2; \vartheta_s)$$

that has the parameter $\vartheta_s \in (0, 1]$. The Gumbel-survival copula density is "mirror-symmetric" to the Gumbel copula density.

Cross-dependence measures

Beside the independent copula that implies no cross-dependence, various copulae imply various cross-dependence structures. Therefore, choosing a suitable copula should be quite important for the parametric-copula-based studies.

Cross-dependence measures

Beside the independent copula that implies no cross-dependence, various copulae imply various cross-dependence structures. Therefore, choosing a suitable copula should be quite important for the parametric-copula-based studies.
To classify and to quantify the cross-dependence structures implied by various copulae, we have to define the associated cross-dependence measures.

Concordance and tail-dependence may be two most cross-dependence structures of a copula.

Concordance

A pair of uniform random variables is said to be concordant (dis-concordant) if their observations tend to cluster around the 45^o (-45^o) line: $u_1 = u_2$ ($u_1 = 1 - u_2$).

Kendall's tau

Instead of using Pearson's correlation coefficient, the copula literature often measures the concordance by using Kendall's tau:

$$\tau = 4 \iint_{[0,1]^2} C(u_1, u_2) \mathrm{d} C(u_1, u_2) - 1. \tag{4}$$

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$$\tau = 4 \iint_{[0,1]^2} C(u_1, u_2) \mathrm{d} C(u_1, u_2) - 1. \tag{4}$$

Similar to the correlation coefficient (ρ), Kendall's tau is always bound in [-1,1]. Its sign represents the direction of concordance (positive for concordance and negative for dis-concordance), and its magnitude indicates the strength of concordance (or dis-concordance).

- It is quite easy to see that C_I implies $\tau = 0$ (no concordance).
- It is also known that for the copulae C_N and C_t,

$$\tau = \frac{2}{\pi} \arcsin(\rho) \tag{5}$$

  is a monotone transformation of ρ.
- For C_G,

$$\tau = 1 - \vartheta$$

  must be non-negative. Therefore, unlike the normal and t copulae, C_G and C_G^s are unable to interpret the structure of dis-concordance.

Quantile-exceedances (Tail events)

The tail-dependence measures assess the probabilities of the lower-u tail event:

$$A_{iL}(u) = \{u_{it} \mid u_{it} < u\}\ , \qquad u \in (0, 0.5],$$

and the upper-u tail event:

$$A_{iU}(u) := \{u_{it} \mid u_{it} \geq u\}\ , \qquad u \in [0.5, 1).$$

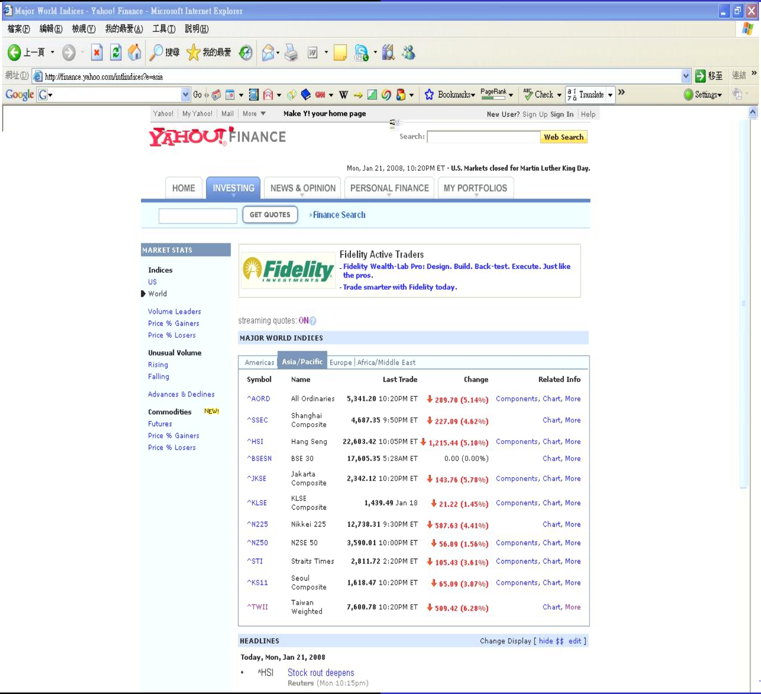
These two tail events are, respectively, equivalent to the following "quantile-exceedance" sets:

$$A_{iL}(u) = \left\{ y_{it} \,\middle|\, y_{it} < F_i^{-1}(u) \right\}, \qquad u \in (0, 0.5],$$

and

$$A_{iU}(u) = \left\{ y_{it} \,\middle|\, y_{it} \geq F_i^{-1}(u) \right\}, \qquad u \in (0, 0.5].$$

In financial applications, unlike Kendall's tau that measures the cross-dependence between markets in "normal" states, these tail-dependence measures can be used to characterize the the cross-dependence between markets at certain "downside" and "upside" states and therefore are particularly useful for risk management.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^SSEC	Shanghai Composite	4,687.35	9:50PM ET	227.09 (4.62%)	Chart, More
^HSI	Hang Seng	22,603.42	10:05PM ET	1,215.44 (5.10%)	Components, Chart, More
^BSESN	BSE 30	17,605.35	5:28AM ET	0.00 (0.00%)	Chart, More
^JKSE	Jakarta Composite	2,342.12	10:20PM ET	143.76 (5.78%)	Components, Chart, More
^KLSE	KLSE Composite	1,439.49	Jan 18	21.22 (1.45%)	Components, Chart, More
^N225	Nikkei 225	12,738.31	9:30PM ET	587.63 (4.41%)	Chart, More
^NZ50	NZSE 50	3,590.01	10:00PM ET	56.89 (1.56%)	Components, Chart, More
^STI	Straits Times	2,811.72	2:20PM ET	105.43 (3.61%)	Components, Chart, More
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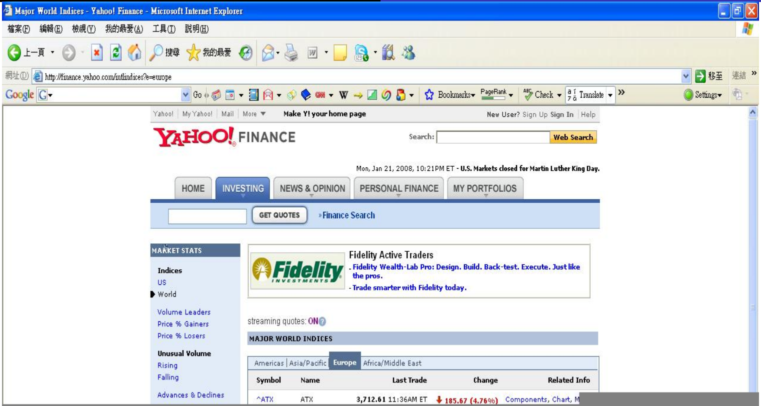
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Given the tail events, we can define the lower-u tail-dependence measure as the conditional probability:

$$\lambda_L(u) := \mathbb{P}(A_{1L}(u)|A_{2L}(u)) = \frac{C(u,u)}{u}, \quad u \in (0, 0.5],$$

and the upper-u tail-dependence measure as the conditional probability:

$$\lambda_U(u) := \mathbb{P}(A_{1U}(u)|A_{2U}(u)) = \frac{C^s(1-u, 1-u)}{1-u}, \qquad u \in [0.5, 1).$$

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The Fréchet-Hoeffding inequality implies that the measures: $\lambda_L(u)$ and $\lambda_U(u)$ are always bounded in [0,1] for any copula C.

The lower-u tail events are independent if, and only if, $\lambda_L(u) = u$; that is,

$$\mathbb{P}(A_{1L}(u)|A_{2L}(u)) = \mathbb{P}(A_{1L}(u)).$$

On the other hand, the upper-u tail events are independent if, and only if, $\lambda_U(u) = 1 - u$.

Given $\lambda_L(u)$ and $\lambda_U(u)$, we can further define the lower "extreme-values" dependence:

$$\lambda_L^* := \lim_{u \to 0^+} \lambda_L(u)$$

and the upper extreme-values dependence:

$$\lambda_U^* := \lim_{u \to 1^-} \lambda_U(u);$$

see, e.g., Joe (1997).

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see, e.g., Joe (1997).
Some studies refer to a copula with $\lambda_L^* = 0$ ($\lambda_U^* = 0$) as a "lower-tail-independent" (an "upper-tail-independent") copula. However, this terminology ignores the fact that the tail events are not the same as the extreme events.

It is known that

1. C_N implies $\lambda_L^* = \lambda_U^* = 0$,
2. C_G implies $\lambda_L^* = 0$ and $0 \leq \lambda_U^* = 2 - 2^{\vartheta} < 1$,
3. C_G^s implies $0 \leq \lambda_L^* = 2 - 2^{\vartheta_s} < 1$ and $\lambda_U^* = 0$,
4. $C_o = C_t$ implies
   $\lambda_L^* = \lambda_U^* = 2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{1-\rho}/\sqrt{1+\rho}\right)$;

see, e.g., Embrechts, Lindskog, and McNeil (2003) and Schmidt (2004).

However, as shown by Figures 1 and 2, $\lambda_L(u) - u$ and $\lambda_U(u) - (1 - u)$ are all positive, if $u \neq 0$ and $u \neq 1$, for C_N, C_G, C_G^s, and C_t. In other words, these copulae are all tail-dependent.

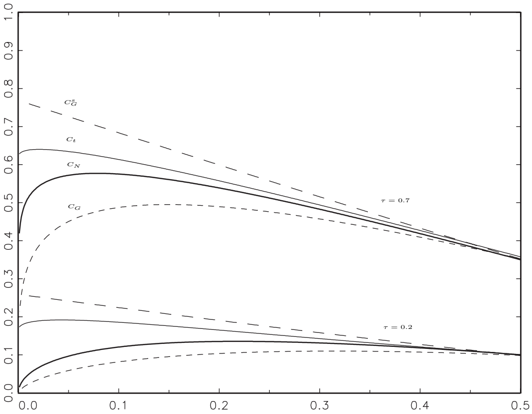

Figure 1. The differences: $\lambda_L(u) - u$ implied by C_N, C_t, C_G, and C_G^s.

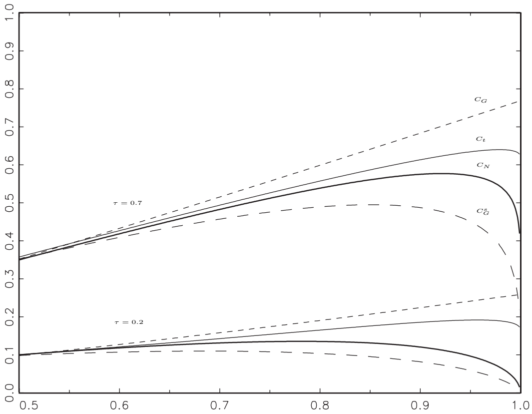

Figure 2. The differences: $\lambda_U(u) - (1-u)$ implied by C_N, C_t, C_G, and C_G^s.

Copula density function

From (2), we can define the copula density function:

$$c(u_1, u_2, \ldots, u_n) := \frac{\partial^n}{\partial u_1 \partial u_2 \ldots \partial u_n} C(u_1, u_2, \ldots, u_n)$$
$$= \frac{f_{\mathbf{y}}(F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_n^{-1}(u_n))}{\prod_{i=1}^{n} f_i(F_i^{-1}(u_i))},$$

where f_i and $f_{\mathbf{y}}$ are, respectively, the probability density function of F_i and $F_{\mathbf{y}}$.

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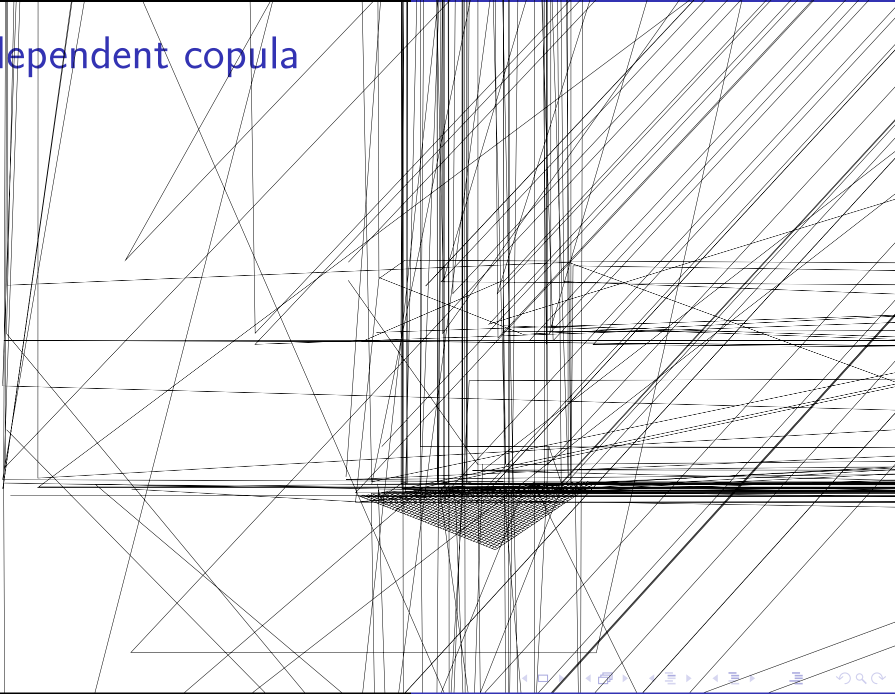
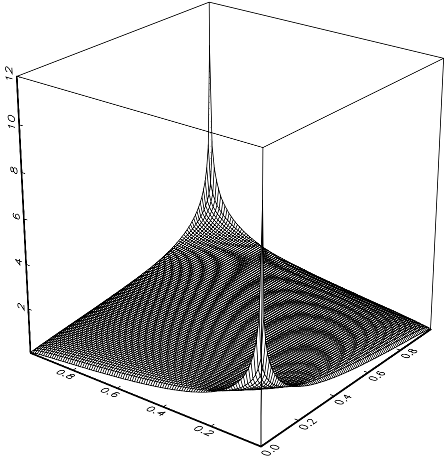
where f_i and $f_{\mathbf{y}}$ are, respectively, the probability density function of F_i and $F_{\mathbf{y}}$.
The independent, normal, Gumbel, t, and any other copula density functions can be easily derived in accordance with this formula.

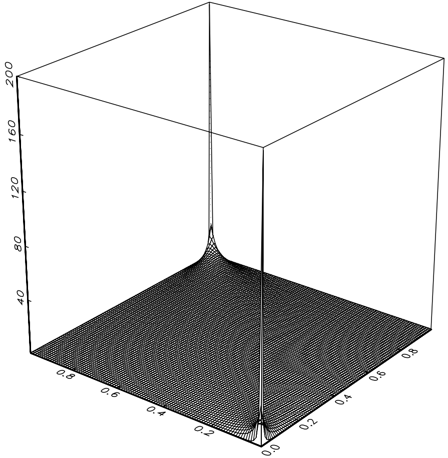
Independent copula



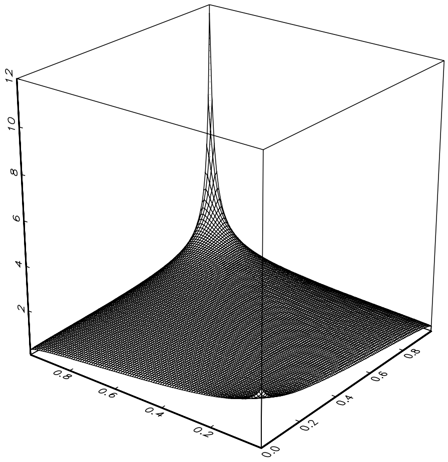
Normal copula (lower concordance, $\tau = 0.2$)



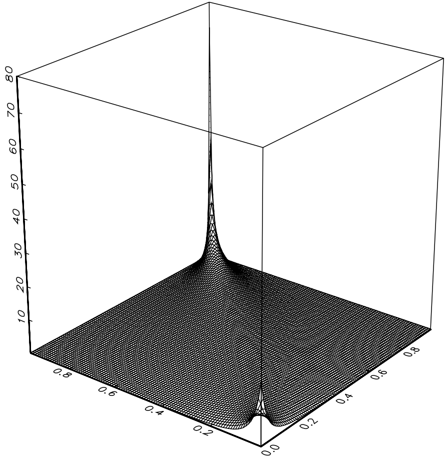
Normal copula (higher concordance, $\tau = 0.7$)



Gumbel copula (lower concordance, $\tau = 0.2$)



Gumbel copula (higher concordance, $\tau = 0.7$)



Characteristics

These figures show that

- C_N and C_G both have a higher density at the 45^o line,
- the clustering tendency of the lower-u (upper-u) tail events increases as $u \to 0^+$ ($u \to 1^-$),
- this clustering tendency increases with the strength of concordance,
- $c_N(u, u; \rho)$ is symmetric to $u = 0.5$,
- $c_G(u, u; \vartheta)$ is asymmetric to $u = 0.5$ and has a heavier upper tail.

The L, U, and J dependence

- In accordance with the shape of c_N and c_G at the 45^o line, Hu (2006) referred to C_N and C_G as copulae with the "U-shaped" and "J-shaped" dependence structures, respectively.
- Because c_G^s is mirror-symmetric to c_G about the line: $u_1 = 1 - u_2$ when $\vartheta = \vartheta_s$. By this mirror-symmetry, it should be understood that C_G^s is a copula with the "L-shaped" dependence structure (heavier lower tail).
- Similar to the normal copula, the t copula also has an "U-shaped" dependence.
- This illustrates that tail-dependence is much more important than concordance in discriminating between different parametric copulae.

The true copula is unknown

In practice, the true copula and the true marginal distributions are unknown. This generates the issues on how to specify copula, how to estimate copula, and how to test copula.

Existing approaches

Under the assumption that $\{\mathbf{y}_t\}$ is an i.i.d. sequence, there is a parametric approach that deals with these issues by

- specifying certain parametric (marginal distributions and) copulae,
- estimating the parameters of (the marginal distributions and) the copulae by the maximum likelihood (ML) method, either in an one-stage way or in a two-stage way,
- evaluating the copula models
  1. the AIC's,
  2. the uniformity of "conditional copulae" (the derivatives of the copula taken with respect to its margins) and the Kolmogorov test or the Pearson χ^2 test.

There is also a semi-parametric approach that replaces the parametric marginal distributions with the empirical distributions.

- Genest et al. (1995) is a well-known paper on this approach that derives the asymptotic variance-covariance matrix of the two-stage (semi-parametric) MLEs.

Stylized facts of data

The above-mentioned parametric and semi-parametric approaches are quite popular in empirical finance. Nevertheless, the i.i.d. assumption of $\{\mathbf{y}_t\}$ is obviously improper for financial data.

- It is well-recognized that stock returns have volatility clustering, leverage effects, and other stylized facts.
- Ignoring such serial dependence structures, the marginal distributions and hence the resulting copula models are unlikely to be correctly specified.
- It is crucial to deal with these problems for analyzing the cross-dependence structures in an adequate way.

Connecting copula and GARCH

Motivated by the above-mentioned problems, Hu (2006), Jondeau and Rockinger (2006), and Patton (2006a,b) suggested replacing the unconditional marginal distributions with the conditional marginal distributions that are based on certain univariate GARCH-type models.

Connecting copula and GARCH

Motivated by the above-mentioned problems, Hu (2006), Jondeau and Rockinger (2006), and Patton (2006a,b) suggested replacing the unconditional marginal distributions with the conditional marginal distributions that are based on certain univariate GARCH-type models.

Their models are special cases of the generalized copula-based multivariate dynamic (CMD) model that we discuss below.

A dynamic location-scale specification

Denote

- I_i^{t-1}: the information set generated by $Y_i^{t-1} := (y_{i,t-1}, y_{i,t-2}, \ldots)$ and some pre-determined variables at time t,
- $\mathbf{I}^{t-1} := (I_1^{t-1}, I_2^{t-1}, \ldots, I_n^{t-1})$.

We base the CMD model on the dynamic location-scale specification:

$$\mathbf{y}_t = \mathbf{m}_t(\mathbf{x}_t, \boldsymbol{\alpha}) + \mathbf{h}_t(\mathbf{x}_t, \boldsymbol{\alpha})^{1/2}\varepsilon_t, \qquad \boldsymbol{\alpha} \in \mathbf{A} \subset \mathbb{R}^a, \tag{6}$$

in which

- $\mathbf{x}_t$ is a vector of $\mathbf{I}^{t-1}$-measurable random variables,
- $\mathbf{m}_t := \mathbf{m}_t(\mathbf{x}_t, \boldsymbol{\alpha})$ is a $n \times 1$ vector with the ith element $m_{it} := m_{it}(\mathbf{x}_t, \alpha_i)$,
- $\mathbf{h}_t := \mathbf{h}_t(\mathbf{x}_t, \boldsymbol{\alpha})$ is a $n \times n$ "diagonal matrix" with the ith diagonal term $h_{it} := h_{it}(\mathbf{x}_t, \alpha_i)$,
- $\boldsymbol{\alpha} := (\alpha_1^\top, \alpha_2^\top, \ldots, \alpha_n^\top)^\top$ is an $a \times 1$ parameters vector in the parameters space $\mathbf{A}$, $\alpha_i \in A_i \subset \mathbb{R}^{a_i}$ is an $a_i \times 1$ parameters vector, and $a = \sum_{i=1}^{n} a_i$;
- $\boldsymbol{\varepsilon}_t := (\varepsilon_{1t}, \varepsilon_{2t}, \ldots, \varepsilon_{nt})^\top$ is the standardized errors vector with $\varepsilon_{it} := h_{it}^{-1/2}(y_{it} - m_{it})$, $\mathbb{E}[\varepsilon_{it}] = 0$, and $\mathrm{var}[\varepsilon_{it}] = 1$.

The marginal models

By combing (6) with the conditional (standardized error) distribution assumption:

- $\varepsilon_{it}|\mathbf{x}_t$ has the conditional distribution $F_{\varepsilon_i}(\cdot|\mathbf{x}_t;\beta_i)$ with a $b_i \times 1$ parameters vector $\beta_i \in B_i \subset \mathbb{R}^{b_i}$ and the conditional probability density function:

$$f_{\varepsilon_i}(\epsilon|\mathbf{x}_t;\beta_i) := \frac{\partial}{\partial \epsilon} F_{\varepsilon_i}(\epsilon|\mathbf{x}_t;\beta_i), \qquad \forall\ \epsilon \in \mathbb{R},$$

we can obtain a set of fully specified marginal models:

$$F_{y_i}(y|\mathbf{x}_t;\gamma_i) := F_{\varepsilon_i}\left(h_{it}^{-1/2}(y-m_{it})\middle|\,\mathbf{x}_t;\beta_i\right), \qquad \forall\ y\in\mathbb{R}, \tag{7}$$

with the parameters vector $\gamma_i := (\alpha_i^\top, \beta_i^\top)^\top \in \Gamma_i \subset \mathbb{R}^{a_i+b_i}$; $i=1,2,\ldots,n$. Hereafter, we also denote $\boldsymbol{\beta} := (\beta_1^\top, \beta_2^\top, \ldots, \beta_n^\top)^\top \in \mathbf{B} \subset \mathbb{R}^b$ and $b := \sum_{i=1}^n b_i$.

The CMD model

By coupling the marginal models with the copula model $C(\cdot|\mathbf{x}_t;\theta)$, a generalized parametric CMD model is derived:

$$F_{\mathbf{y}}(\mathbf{y}|\mathbf{x}_t;\lambda) := C\left(F_{y_1}(y_1|\mathbf{x}_t;\gamma_1),\ldots,F_{y_n}(y_n|\mathbf{x}_t;\gamma_n)\right|\mathbf{x}_t;\theta\Big), \tag{8}$$

where $\lambda := (\boldsymbol{\gamma}^\top,\theta^\top)^\top$ is a $(a+b+r)\times 1$ vector of parameters with $\boldsymbol{\gamma} := (\boldsymbol{\alpha}^\top,\boldsymbol{\beta}^\top)^\top$.

The constant conditional correlation (CCC) model of Bollerslev (1990) and the dynamic conditional correlation (DCC) models of Engle (2002) and Tse and Tsui (2002) are also special case of this CMD model where the F_{y_i}'s and C are both "normal".

The key feature of this CMD model is that the parameter vectors γ_i's are separable for different i's. This permits us to present the marginal models as a set of univariate GARCH-type models conditional on the same information set $\mathbf{I}^{t-1}$. Accordingly, we can estimate the γ_i's separately before the copula analysis.

As demonstrated by Bauwens, Laurent, and Rombouts (2006, Section 2.3), the CCC (or DCC) model and the copula-based models of Jondeau and Rockinger (2006) and Patton (2006a) are in the same sub-class of multivariate GARCH-type models obtained by certain nonlinear combinations of univariate GARCH-type models. This interpretation applies to the generalized CMD model (94).

In comparison, the VEC model of Bollerslev, Engle, and Wooldridge (1988) and the BEKK model of Engle and Kroner (1995) are other types of multivariate GARCH-type models that may not have completely separable parameters for various i's.

This difference makes the CCC and DCC models much easier to estimate than the VEC and BEKK models; see, e.g., Engle and Sheppard (2001), Engle (2002), and Tse and Tsui (2002). In addition to this advantage, the CMD model can also be flexibly applied to explore the cross-dependence structures using various C's.

Dynamic copulae

The static copulae can be easily extended to being the dynamic copulae by re-specifying their parameters as certain dynamic functions of $\mathbf{x}_t$, as in Jondeau and Rockinger (2006) and Patton (2006a).

By replacing the CCC coefficient ρ of $C_N(u_1, u_2; \rho)$ with the DCC coefficient $\rho_t = \rho_t(\mathbf{x}_t; \theta)$, such as that of Tse and Tsui (2002):

$$\rho_t = (1-\kappa_1-\kappa_2)\kappa_o+\kappa_1\rho_{t-1}+\kappa_2\frac{\sum_{k=1}^{m}\varepsilon_{1,t-k}\varepsilon_{2,t-k}}{\sqrt{\left(\sum_{k=1}^{m}\varepsilon_{1,t-k}^2\right)\left(\sum_{k=1}^{m}\varepsilon_{2,t-k}^2\right)}}, \tag{9}$$

where $-1 \leq \kappa_o \leq 1$, $0 \leq \kappa_1 \leq 1$, $0 \leq \kappa_2 \leq 1$, $\kappa_1 + \kappa_2 \leq 1$, and $m = 2$, we can define the dynamic normal copula $C_N(u_1, u_2|\mathbf{x}_t; \theta)$.

By using the dynamic parameters $\vartheta_t = 1 - \frac{2}{\pi}\arcsin(\rho_t)$ and $\vartheta_{s,t} = 1 - \frac{2}{\pi}\arcsin(\rho_t)$ in place of the parameters ϑ and ϑ_s of the static C_G and C_G^s, we can also define the dynamic Gumbel copula $C_G(u_1, u_2|\mathbf{x}_t;\theta)$ and the dynamic Gumbel-survival copula $C_G^s(u_1, u_2|\mathbf{x}_t;\theta)$, respectively.

Similarly, we can define the dynamic t copula $C_t(u_1, u_2|\mathbf{x}_t; \theta)$ by using the same ρ_t to replace the CCC coefficient ρ of $C_t(u_1, u_2; \rho, \nu)$.

By fixing the same ρ_t, these dynamic copulae have the same dynamic Kendall's tau $\tau(\mathbf{x}_t;\theta)$. However, they have different tail-dependence structures characterized by the dynamic lower-u tail-dependence measures

$$\lambda_L(u|\mathbf{x}_t;\theta) = \frac{1}{u} C(u, u|\mathbf{x}_t;\theta)$$

and the dynamic upper-u tail-dependence measures

$$\lambda_U(u|\mathbf{x}_t;\theta) = \frac{1}{1-u} C^s(1-u, 1-u|\mathbf{x}_t;\theta).$$

The conditional Sklar's theorem

Denote

- $F^o_{y_i}(\cdot|\mathbf{I}^{t-1})$: the true conditional distribution of $y_{it}|\mathbf{I}^{t-1}$,
- $F^o_{\mathbf{y}}(\cdot|\mathbf{I}^{t-1})$: the true conditional multivariate distribution of $\mathbf{y}_t|\mathbf{I}^{t-1}$.

The conditional Sklar theorem by Patton (2006a, IER):

- there exists a unique conditional copula $C_o(\cdot|\mathbf{I}^{t-1}) : [0,1]^n \to [0,1]$ such that

$$F^o_{\mathbf{y}}(\mathbf{y}|\mathbf{I}^{t-1}) = C_o\left(F^o_{y_1}(y_1|\mathbf{I}^{t-1}), \ldots, F^o_{y_n}(y_n|\mathbf{I}^{t-1}) \middle| \, \mathbf{I}^{t-1} \right), \tag{10}$$

  for all $\mathbf{y} := (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

Importantly, the univariate conditional distributions and the conditional copula must have the “same” information set as the multivariate conditional distribution. This is why we have to specify the same information set $\mathbf{I}^{t-1}$ for all the marginal models and the copula model.

Specification correctness conditions

By comparing the CMD model:

$$F_{\mathbf{y}}(\mathbf{y}|\mathbf{x}_t;\lambda) := C\left(F_{y_1}(y_1|\mathbf{x}_t;\gamma_1),\ldots,F_{y_n}(y_n|\mathbf{x}_t;\gamma_n)\right|\mathbf{x}_t;\theta\right),$$

with the conditional Sklar's theorem:

$$F^o_{\mathbf{y}}(\mathbf{y}|\mathbf{I}^{t-1}) = C_o\left(F^o_{y_1}(y_1|\mathbf{I}^{t-1}),\ldots,F^o_{y_n}(y_n|\mathbf{I}^{t-1})\right|\mathbf{I}^{t-1}\right),$$

it is clear that the CMD model is correctly specified for the true conditional distribution $F^o_{\mathbf{y}}(\cdot|\mathbf{I}^{t-1})$, if

1. the marginal models are correctly specified in the sense that there exists some (unique) vector $\gamma_{io} := (\alpha_{io}^\top, \beta_{io}^\top)^\top \in \Gamma_i$ at which

$$F_{y_i}(\cdot|\mathbf{x}_t; \gamma_{io}) = F_{y_i}^o(\cdot|\mathbf{I}^{t-1}), \qquad \forall\ i = 1, 2, \ldots, n,$$

2. the copula model is correctly specified in the sense that

$$C(\cdot|\mathbf{x}_t; \theta_o) = C_o(\cdot|\mathbf{I}^{t-1}), \tag{11}$$

   for some (unique) $\theta_o \in \Theta$.

In this study, we treat Condition 1 as a maintained assumption, denoted as assumption [A], and focus on proposing copula tests for Condition 2.

The ML methods

Let $c(\cdot|\mathbf{x}_t;\theta)$ be the density function of $C(\cdot|\mathbf{x}_t;\theta)$. The parameters of the CMD model may be estimated using different ML methods.

The one-stage ML method

$$\hat{\lambda}_T = \mathrm{argmax}_{\lambda} \frac{1}{T} \sum_{t=1}^{T} \ln c \left(F_{y_1}(y_1|\mathbf{x}_t; \gamma_1), \ldots, F_{y_n}(y_n|\mathbf{x}_t; \gamma_n) \middle| \mathbf{x}_t; \theta \right).$$

The two-stage ML method

1. the 1st stage:

$$\hat{\gamma}_{iT} = \operatorname{argmax}_{\gamma_i} \frac{1}{T} \sum_{t=1}^{T} \ln f_{y_i}(y_i | \mathbf{x}_t; \gamma_i),$$

2. the 2nd stage:

y t

$$\hat{\theta}_T = \operatorname{argmax}_{\theta} \frac{1}{}$$

1

γ_i

The three-stage ML method

1. the 1st stage:

$$\hat{\alpha}_{iT} = \mathrm{argmax}_{\alpha_i} - \frac{1}{2}\ln 2\pi - \frac{1}{2T}\sum_{t=1}^{T}\ln h_{it} - \frac{1}{2T}\sum_{t=1}^{T} h_{it}^{-1}(y_{it} - m_{it})^2;$$

2. the 2nd stage:

$$\hat{\beta}_{iT} = \mathrm{argmax}_{\beta_i}\frac{1}{T}\sum_{t=1}^{T}\ln f_{\varepsilon_i}(\hat{\varepsilon}_{it}|\mathbf{x}_t;\beta_i), \quad \hat{\varepsilon}_{it} := \varepsilon_{it}|_{\alpha=\hat{\alpha}_{iT}}$$

3. the 3rd stage:

$$\hat{\theta}_T = \mathrm{argmax}_{\theta}\frac{1}{T}\sum_{t=1}^{T}\ln c\left(F_{y_1}(y_1|\mathbf{x}_t;\hat{\gamma}_1),\ldots,F_{y_n}(y_n|\mathbf{x}_t;\hat{\gamma}_n)\middle|\,\mathbf{x}_t;\theta\right).$$

Compared to the one-stage method, the multi-stage estimation methods are much easier to implement when the CMD model is complicated. Moreover, the latter is also consistent with the "bottom-up" model-building procedure which is quite important for building a fully specified parametric model in a logically consistent way; see Wooldridge (1991, JE).

The asymptotic variance-covariance matrix

These asymptotic variance-covariance matrices of the two-stage and three-stage MLEs are quite different from that of the one-stage MLEs. It should be careful in estimating the asymptotic variance-covariance matrix and computing the test statistics when a multi-stage estimation method is used. See Patton (2006b, JAE) for the case of the two-stage MLEs.

The three-stage MLEs

In the paper, we show that

- $\sqrt{T}(\hat{\alpha}_{iT} - \alpha_{io}) \xrightarrow{d} N(0, \Sigma^o_{\alpha i}), \qquad \Sigma^o_{\alpha i} := \mathbb{E}[\psi^o_{\alpha,it}\psi^{o\top}_{\alpha,it}],$
- $\sqrt{T}(\hat{\beta}_{iT} - \beta_{io}) \xrightarrow{d} N(0, \Sigma^o_{\beta i}), \qquad \Sigma^o_{\beta i} := \mathbb{E}[\psi^o_{\beta,it}\psi^{o\top}_{\beta,it}],$
- $\sqrt{T}(\hat{\theta}_{T} - \theta_{o}) \xrightarrow{d} N(0, \Sigma^o_{\theta}), \qquad \Sigma^o_{\theta} := \mathbb{E}[\psi^o_{\theta t}\psi^{o\top}_{\theta t}],$

The margingale-difference sequences

$$\psi_{\alpha,it} = \left\{ \mathbb{E}[w_{it} w_{it}^{\top}] + \frac{1}{2}\mathbb{E}[z_{it} z_{it}^{\top}] \right\}^{-1} \left\{ w_{it}\varepsilon_{it} + \frac{1}{2} z_{it}(\varepsilon_{it}^2 - 1) \right\};$$

$$\psi_{\beta,it} = \mathbb{E}[l_{\beta,it} l_{\beta,it}^{\top}]^{-1} \left\{ l_{\beta,it} + \zeta_{it}^{o} \psi_{\alpha,it} \right\},$$

$$\psi_{\theta t} = \mathbb{E}[l_{\theta t} l_{\theta t}^{\top}]^{-1} \left\{ l_{\theta t} + \sum_{i=1}^{n} \left(\xi_{\alpha i}^{o} \psi_{\alpha,it} + \xi_{\beta i}^{o} \psi_{\beta,it} \right) \right\}.$$

See also the Appendix of my paper for the notations.

Testing

Because the cross-dependence structures of $\mathbf{y}_t|\mathbf{I}^{t-1}$ are fully characterized by the true, but unknown, copula function $C_o(\cdot|\mathbf{I}^{t-1})$, it is quite important to test the adequacy of $C(\cdot|\mathbf{x}_t;\theta)$ in copula studies. Beside Chen, Fan, and Patton (2004, working paper) that proposed a non-parametric test, we find no other formal tests for the CMD model before this study. Our tests are established in a parametric way.

The conditional PITs

Denote the conditional PIT:

$$u_{it} := F_{y_i}(y_{it}|\mathbf{x}_t; \gamma_i)$$

and the PIT vectors:

$$\mathbf{u}_t := (u_{1t}, u_{2t} \ldots, u_{nt})^\top.$$

Given assumption [A], $\mathbf{u}_{ot} := (u^o_{1t}, u^o_{2t}, \ldots, u^o_{nt})^\top$ is a $n \times 1$ vector of $U(0,1)$ random variables, where $u^o_{it} := u_{it}|_{\gamma_i = \gamma_{io}}$.

The test functions

Let $\phi_t := \phi(\mathbf{u}_t|\mathbf{x}_t;\theta)$ be a $q \times 1$ vector of testing indicators. Denote $\gamma_o := (\boldsymbol{\alpha}_o^\top, \boldsymbol{\beta}_o^\top)^\top$, $\lambda_o := (\boldsymbol{\gamma}_o^\top, \theta_o^\top)^\top$, and $\phi_{ot} := \phi_t|_{\lambda=\lambda_o}$.

The test functions

Let $\phi_t := \phi(\mathbf{u}_t|\mathbf{x}_t;\theta)$ be a $q \times 1$ vector of testing indicators. Denote $\gamma_o := (\boldsymbol{\alpha}_o^\top, \boldsymbol{\beta}_o^\top)^\top$, $\lambda_o := (\boldsymbol{\gamma}_o^\top, \theta_o^\top)^\top$, and $\phi_{ot} := \phi_t|_{\lambda=\lambda_o}$. Suppose that the condition

$$\mathbb{E}[\phi_{ot}|\mathbf{I}^{t-1}] = 0$$

is satisfied under the null hypothesis.

For example, if the copula model has the same Kendall's tau as the true copula, then the condition $\mathbb{E}[\phi_{ot}|\mathbf{I}^{t-1}] = 0$ is satisfied for the following ϕ:

$$\phi_\tau(\mathbf{u}_t|\mathbf{x}_t;\theta) = 4C(\mathbf{u}_t|\mathbf{x}_t;\theta) - 1 - \tau(\mathbf{x}_t;\theta). \qquad (12)$$

If the copula model has the same upper-u dependence structure as the true copula, then the condition $\mathbb{E}[\phi_{ot}|\mathbf{I}^{t-1}] = 0$ is satisfied for

$$\phi_{U(u)}(\mathbf{u}_t|\mathbf{x}_t;\theta) = \frac{1}{1-u}[I(u_{1t} \geq u)I(u_{2t} \geq u)] - \lambda_U(u|\mathbf{x}_t;\theta),$$
$$u \in [0.5, 1), \tag{13}$$

in which the indicator function: $I(\epsilon \geq \epsilon_o) = 1$ if $\epsilon \geq \epsilon_o$ and $I(\epsilon \geq \epsilon_o) = 0$ if $\epsilon < \epsilon_o$, where $\epsilon, \epsilon_o \in \mathbb{R}$.

If the copula model has the same lower-u dependence structure as the true copula, then the condition $\mathbb{E}[\phi_{ot}|\mathbf{I}^{t-1}] = 0$ is satisfied for

$$\phi_{L(u)}(\mathbf{u}_t|\mathbf{x}_t;\theta) = \frac{1}{u} I(u_{1t} < u) I(u_{2t} < u) - \lambda_L(u|\mathbf{x}_t;\theta), \quad u \in (0, 0.5]. \tag{14}$$

We may also set the following $2p$-dimensional ϕ:

$$\phi_{LU} := (\phi_{L(v_1)}, \ldots, \phi_{L(v_p)}, \phi_{U(1-v_p)}, \ldots, \phi_{U(1-v_1)})^\top, \qquad (15)$$

for some $v_i \in (0, 0.5)$, $v_i < v_{i+1}$, and $i = 1, 2, \ldots, p$, to check the different tail-dependence measures at the same time.

In what follows, we refer to the moment test with $\phi = \phi_\tau$, $\phi_{L(u)}$, $\phi_{U(u)}$, and ϕ_{LU} as the M_τ test, the $M_L(u)$ test, the $M_U(u)$ test, and the M_{LU} test respectively. The M_τ test is a concordance test, and the $M_L(u)$, $M_U(u)$, and M_{LU} tests are tail-dependence tests.

Generalized first-order asymptotics

Our test checks the condition $\mathbb{E}[\phi_{ot}] = 0$ by examining whether the statistic

$$\hat{D}_T := T^{-1} \sum_{t=1}^{T} \hat{\phi}_t,$$

where $\hat{\phi}_t := \phi_t|_{\lambda=\hat{\lambda}_T}$, is significantly different from zero.

If ϕ is twice continuously differentiable, then we can apply the standard Taylor expansion to show that

$$\sqrt{T}\hat{D}_T = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\phi_{ot} + \left[\frac{1}{T}\sum_{t=1}^{T}\nabla_{\theta^\top}\phi_t\right]_{\lambda=\lambda_o}\sqrt{T}(\hat{\theta}_T - \theta_o)$$

$$+\sum_{i=1}^{n}\left\{\left[\frac{1}{T}\sum_{t=1}^{T}p_{it}\nabla_{\gamma_i^\top}u_{it}\right]_{\lambda=\lambda_o}\sqrt{T}(\hat{\gamma}_{iT} - \gamma_{io})\right\} + o_p(1), \quad (16)$$

where $p_{it} := \frac{\partial}{\partial u_{it}}\phi_t$.

Following the generalized first-order asymptotics of Phillips (1991, ET), this result may also be applied to the ϕ's that are composed of the indicator function, such as $\phi_{L(u)}$ and $\phi_{U(u)}$. This is due to the fact that, although the indicator function is not differentiable in the ordinary sense, it is "differentiable" in the following sense

$$\frac{\partial}{\partial \epsilon} I(\epsilon \geq \epsilon_o) = \delta(\epsilon - \epsilon_o),$$

where δ represents the Dirac delta function.

Dirac delta

The Dirac delta function is a generalized function that can be understood as the limit of a delta sequence, such as the limit of the $N(\epsilon_o, \sigma^2)$ probability density functions sequence as $\sigma^2 \longrightarrow 0^+$. That is, it may be viewed as the "density function" of a degenerated distribution.

This function has the sifting property (or said the reproducing property):

$$\int_{\mathbb{R}} \delta(\epsilon - \epsilon_o)\mu(\epsilon)\mathrm{d}\epsilon = \mu(\epsilon_o), \tag{17}$$

where μ denotes a continuous function; see Gelfand and Shilov (1964), Bracewell (1999), and Kanwal (2004), among others.

The asymptotics of the concordance test is derived from (16). The asymptotic null distributions of the tail-dependence tests are derived from (16) and (17). See Section 3.2 for an application of the sifting property.

The moment test

The proposed test statistic is of the form:

$$M_T := T\hat{D}_T^\top \hat{\Omega}_T^{-1} \hat{D}_T$$

where $\hat{\Omega}_T$ is a consistent estimator of the asymptotic variance-covariance matrix of $T^{1/2}\hat{D}_T$:

$$\Omega_o = \mathbb{E}[\varphi_{ot}\varphi_{ot}^\top],$$

$$\varphi_{ot} := \phi_{ot} + \eta_{co}\psi_{\theta t}^o + \sum_{i=1}^{n} \eta_{io}\psi_{\gamma,it}^o, \qquad \psi_{\gamma,it}^o := (\psi_{\alpha,it}^{o\top}, \psi_{\beta,it}^{o\top})^\top;$$

see Section 2 for the undefined notations.

This test statistic has the standard asymptotic null distribution:

$$M_T \xrightarrow{d} \chi^2(q).$$

In the case where $q = 1$, we can also express the M test statistic as $M'_T = \sqrt{T}\hat{D}_T/\hat{\Omega}_T^{1/2}$. This statistic has the asymptotic null distribution $N(0,1)$, and its sign may contain some useful information about the discrepancy between the true and postulated cross-dependence structures.

Simulation

- The marginal models $F_{y_i}(y_i|\mathbf{x}_t;\gamma_i)$ are specified to have the AR(1) conditional mean $m_{it} = \alpha_{mo} + \alpha_{m1} y_{i,t-1}$, the GARCH(1,1) conditional variance $h_{it} = \alpha_{ho} + \alpha_{h1} h_{i,t-1} + \alpha_{h2}(y_{i,t-1} - m_{i,t-1})^2$, and the i.i.d. $N(0,1)$ standardized error ε_{it} for both $i = 1, 2$.
- The marginal models are set to be correctly specified in the sense of assumption [A] with $(\alpha_{mo}, \alpha_{m1}, \alpha_{ho}, \alpha_{h1}, \alpha_{h2}) = (0.01, 0.05, 0.05, 0.85, 0.1)$.

- The CMD model being tested is of the form:

$$F_{\mathbf{y}}(\mathbf{y}|\mathbf{x}_t;\lambda) = C\left(F_{y_1}(y_1|\mathbf{x}_t;\gamma_1), F_{y_2}(y_2|\mathbf{x}_t;\gamma_2);\theta\right).$$

- The copula model: $C = C_I$ and the static C_N.
- The true copula: C_I, C_N, C_G, and C_t with the static parameter $\rho = 0.1$ and 0.5 (or equivalently the static Kendall's tau $\tau_1 := 0.0638$ and $\tau_2 := 0.3317$), where C_t has the degrees of freedom $\nu = 4$.
- Sample size: $T = 500, 1000, 2500$, 5% nominal level, one thousand replications.
- The M_τ test, the M_{LU} test with $(v_1, v_2, v_3, 1-v_3, 1-v_2, 1-v_1) = (0.1, 0.3, 0.5, 0.5, 0.7, 0.9)$, and the associated $M_{L(u)}$ and $M_{U(u)}$ tests.

The M_τ test

Table: Empirical sizes and powers of the M_τ test.

	$H_o : C_o = C_I$			$H_o : C_o = C_N$		
C_o	T =500	1000	2500	T =500	1000	2500
C_I	**8.1**	**7.4**	**6.6**	**7.0**	**6.8**	**7.9**
C_N (τ_1)	43.8	68.7	95.8	**6.8**	**8.5**	**7.6**
C_N (τ_2)	100.0	100.0	100.0	**7.6**	**8.4**	**7.5**
C_G (τ_1)	43.4	70.8	96.6	7.5	6.7	8.3
C_G (τ_2)	100.0	100.0	100.0	7.3	9.3	7.2
C_t (τ_1)	38.2	61.2	94.6	6.4	8.1	5.8
C_t (τ_2)	100.0	100.0	100.0	7.8	9.2	9.4

Notes: The bold entries represent the empirical sizes in percentages, and the others are the empirical powers in percentages.

The $M_{L(u)}$, $M_{U(u)}$, and M_{LU} tests

Table: Empirical sizes of the tail-dependence tests.

		$H_o : C_o = C_I$			$H_o : C_o = C_N$		
C_o	test	T =500	1000	2500	T =500	1000	2500
C_I	M_{LU}	**10.9**	**6.6**	**5.6**	**8.2**	**8.1**	**6.5**
	$M_L(0.1)$	**10.1**	**6.2**	**6.0**	**7.4**	**6.0**	**5.5**
	$M_L(0.3)$	**5.5**	**4.7**	**5.4**	**5.8**	**4.3**	**5.9**
	$M_L(0.5)$	**6.5**	**5.7**	**5.7**	**5.6**	**5.2**	**4.9**
	$M_U(0.5)$	**5.2**	**5.9**	**4.6**	**6.5**	**5.2**	**5.6**
	$M_U(0.7)$	**5.5**	**6.6**	**5.8**	**4.2**	**4.2**	**5.2**
	$M_U(0.9)$	**11.6**	**6.3**	**5.0**	**7.6**	**6.6**	**4.8**

Table: Empirical sizes and powers of the tail-dependence tests.

C_o	test	$H_o : C_o = C_I$ T =500	1000	2500	$H_o : C_o = C_N$ T =500	1000	2500
C_N (τ_1)	M_{LU}	19.1	38.1	86.0	**7.3**	**7.1**	**5.6**
	$M_L(0.1)$	8.4	12.1	34.3	**5.4**	**4.5**	**5.1**
	$M_L(0.3)$	20.5	34.2	70.6	**5.0**	**5.0**	**5.0**
	$M_L(0.5)$	21.2	36.7	69.1	**5.2**	**6.1**	**5.7**
	$M_U(0.5)$	17.6	32.1	68.1	**5.1**	**6.2**	**5.8**
	$M_U(0.7)$	18.8	33.7	69.9	**5.5**	**6.0**	**5.1**
	$M_U(0.9)$	9.5	13.1	33.6	**5.7**	**6.6**	**5.2**
C_N (τ_2)	M_{LU}	100.0	100.0	100.0	**6.2**	**6.3**	**4.9**
	$M_L(0.1)$	96.8	99.9	100.0	**6.9**	**5.3**	**3.8**
	$M_L(0.3)$	100.0	100.0	100.0	**5.8**	**5.6**	**4.9**
	$M_L(0.5)$	100.0	100.0	100.0	**5.3**	**5.6**	**5.8**
	$M_U(0.5)$	100.0	100.0	100.0	**5.0**	**6.0**	**5.4**
	$M_U(0.7)$	100.0	100.0	100.0	**4.0**	**5.9**	**6.5**
	$M_U(0.9)$	96.1	100.0	100.0	**5.2**	**5.5**	**5.0**

Table: Empirical powers of the tail-dependence tests.

C_o	test	$H_o : C_o = C_I$ $T=500$	1000	2500	$H_o : C_o = C_N$ $T=500$	1000	2500
C_G (τ_1)	M_{LU}	24.6	52.6	97.1	8.4	12.8	25.9
	$M_L(0.1)$	7.1	7.1	18.8	9.8	8.6	10.8
	$M_L(0.3)$	12.7	23.8	52.0	6.5	9.5	10.6
	$M_L(0.5)$	18.6	33.4	65.2	4.7	5.4	5.5
	$M_U(0.5)$	18.3	31.8	65.1	5.3	6.3	6.0
	$M_U(0.7)$	29.9	52.0	88.3	5.9	7.8	9.2
	$M_U(0.9)$	24.5	48.5	92.8	9.5	16.5	42.9
C_G (τ_2)	M_{LU}	100.0	100.0	100.0	45.4	79.3	99.8
	$M_L(0.1)$	77.5	97.5	100.0	19.9	34.5	70.6
	$M_L(0.3)$	100.0	100.0	100.0	15.9	23.2	47.7
	$M_L(0.5)$	100.0	100.0	100.0	4.6	5.9	5.7
	$M_U(0.5)$	100.0	100.0	100.0	5.7	5.9	6.2
	$M_U(0.7)$	100.0	100.0	100.0	18.4	31.1	65.2
	$M_U(0.9)$	100.0	100.0	100.0	50.4	85.0	99.5

Table: Empirical powers of the tail-dependence tests.

C_o	test	$H_o : C_o = C_I$ T =500	1000	2500	$H_o : C_o = C_N$ T =500	1000	2500
C_t (τ_1)	M_{LU}	43.7	88.7	100.0	22.8	61.5	99.1
	$M_L(0.1)$	40.3	72.7	99.2	17.9	38.0	84.8
	$M_L(0.3)$	31.4	54.3	93.0	8.6	8.3	14.9
	$M_L(0.5)$	23.1	35.0	70.7	7.1	6.1	5.8
	$M_U(0.5)$	18.2	31.4	67.0	5.5	6.2	5.8
	$M_U(0.7)$	31.2	53.5	92.6	7.8	10.0	16.1
	$M_U(0.9)$	41.7	71.6	98.9	18.7	41.2	84.6
C_t (τ_2)	M_{LU}	100.0	100.0	100.0	19.2	37.3	89.2
	$M_L(0.1)$	99.4	100.0	100.0	14.1	24.5	63.6
	$M_L(0.3)$	100.0	100.0	100.0	7.2	8.9	14.2
	$M_L(0.5)$	100.0	100.0	100.0	6.4	5.0	7.6
	$M_U(0.5)$	100.0	100.0	100.0	5.0	5.9	5.5
	$M_U(0.7)$	100.0	100.0	100.0	7.3	8.5	15.3
	$M_U(0.9)$	99.7	100.0	100.0	14.2	27.0	61.8

Main results

- The empirical sizes are close to the 5% nominal level for all the continuously differentiable the discrete test functions, provided that the sample size is suffciiently large.
- In testing C_N against $C_o = C_G$, we see a "J-shaped" power performance which is consistent with the dissimilarity between the J-shaped dependence of $C_o = C_G$ and the U-shaped dependence of C_N.
- In testing C_N against $C_o = C_t$, we see an "U-shaped" power performance which is consistent with the symmetric lower- and upper-extreme-values dependence of $C_o = C_t$ that cannot be interpreted by C_N.

Main results

- This shows that the proposed tail-dependence tests are quite useful in shedding light on the possible directions of copula mis-specification.
- This property is important because C_o is unknown in practical applications and we have to identify the possible causes of mis-specification before re-specifying the mis-specified copula model.
- This simulation also shows that the information of tail-dependence is much more important than the concordance in discriminating between competing copula models.

Empirical study

- In this empirical study, we apply the concordance and tail-dependence tests to explore stock market relationships.
- Copula models: bivariate C_N, C_G, C_G^s, and C_t (and trivariate C_N and C_t).
- If the true copula has the L-shaped (J-shaped) dependence, then the cross-dependence of downside markets is stronger (weaker) than that of the upside markets.

Designs and issues

- By contrast, if the true copula is of the U-shaped dependence, then there will be no such asymmetry.
- This asymmetry (symmetry) is conceptually very close to the correlation asymmetry (symmetry), studied by Longin and Solnik (2001) and Ang and Chen (2002), which is known to have important implications for portfolio diversification and risk management.

Data

- Stock index (P_{it} in the local currency):
  1. the U.S.: Standard & Poor 500 (SP)
  2. the U.S.: Russell 2000 (RS)
  3. the U.K.: Financial Times Stock Exchange 100 (FT)
  4. France: Compagnie des Agents de Change 40 (CA)
  5. Japan: Nikkei 225 (NK)
  6. Hong Kong: Hang Seng (HS)
  7. Taiwan: Taiwan weighted (TW)

- Sampling period: January 1, 1995 – December 31, 2003.
- We consider the daily returns:

$$y_{it} = 100 \times (\ln P_{it} - \ln P_{i,t-1}),$$

where t denotes the t-th "common" calendar trading date of these markets in the sample. The sample size is $T = 1915$.

- Twenty-one pairs of returns $\mathbf{y}_t = (y_{1t}, y_{2t})$:
  1. the U.S. returns: SP-RS,
  2. the U.S.-European returns: SP-FT, SP-CA, RS-FT, RS-CA,
  3. the European returns: FT-CA,
  4. the Asian returns: NK-HS, NK-TW, and HS-TW,
  5. the U.S.-Asian returns: SP-NK, SP-HS, SP-TW, RS-NK, RS-HS, and RS-TW,
  6. the European-Asian returns: FT-NK, FT-HS, FT-TW, CA-NK, CA-HS, and CA-TW.

Please see my paper, entitled "Moment-based Copula Tests for Financial Returns," for the tables that are not reported here.

Building the marginal models

1. Test the serial independence of $\{y_{it}\}$ against serial correlation, volatility clustering, and time irreversibility.
   - Most return series are likely to be serially correlated, volatility-clustered, and time irreversible. The case of NK is the only exception that has volatility clustering but serial uncorrelatedness and time reversibility.

2. Fitting the AR-GARCH and AR-EGARCH models of "$y_{it}|I_i^{t-1}$" and diagnosing these models by using the "estimation-effect-corrected" serial correlation, volatility clustering, and time irreversibility tests, as in Chen (2003, 2007).
    - NK**1** is the only case that has the GARCH specification and the other cases all have the EGARCH specification.
    - The diagnostic tests accept that these GARCH-type models can successfully interpret the serial dependence of $y_{it}|I_i^{t-1}$ for all the return series.

3. Extending the "$y_{it}|I_i^{t-1}$" model to being the "$y_{it}|\mathbf{I}^{t-1}$' 'model.
   - This extension is quite important but often overlooked by practitioners. Patton (2006, IER) is an important exception that emphasizes the role of this extension in applying the conditional Sklar's theorem.
   - For the return combination (y_{it}, y_{jt}), we first check whether the conditional mean and variance of $y_{it}|I_i^{t-1}$ is correlated to I_j^{t-1} by using the causality tests of Cheung and Ng (1996, JE).

- Then, we extend the "$y_{it}|I_i^{t-1}$" model by adding suitable $y_{j,t-k}$'s (and $y_{j,t-k}^2$'s) into the conditional mean (and variance) of the original model.
- Finally, we accept the "$y_{it}|\mathbf{I}^{t-1}$" models that cannot be rejected by the serial correlation, volatility clustering, time irreversibility, and causality tests.
- Importantly, recall that $\mathbf{I}^{t-1} = (I_i^{t-1}, I_j^{t-1})$, therefore the "$y_{it}|\mathbf{I}^{t-1}$" model may change with different I_j^{t-1} being considered. This is likely one of the most complicated things for a correct copula analysis in empirical finance.

The marginal models

The selected "$y_{it}|\mathbf{I}^{t-1}$" models:

1. the U.S. returns: SP**1**-RS**1**,
2. the U.S.-European returns: SP**1**-FT**2**, SP**1**-CA**2**, RS**1**-FT**3**, RS**1**-CA**3**,
3. the European returns: FT**1**-CA**1**,
4. the Asian returns: NK**1**-HS**1**, NK**1**-TW**1**,
5. the U.S.-Asian returns: SP**1**-NK**2**, SP**1**-HS**2**, SP**1**-TW**2**, RS**1**-NK**3**, RS**2**-HS**3**, and RS**1**-TW**3**,
6. the European-Asian returns: FT**1**-NK**4**, FT**1**-HS**4**, FT**1**-TW**4**, CA**1**-NK**5**, CA**1**-HS**5**, and CA**1**-TW**5**.
7. The combination (HS, TW) is the only exception that we cannot find the suitable $y_{it}|\mathbf{I}^{t-1}$ model's.

- To complete the fully specified $y_{it}|\mathbf{I}^{t-1}$ models, we assume that the standardized errors of these models have the (fixed) skewed t distributions, see Hansen (1994). This assumption is accepted by Bai's (2003) distribution test.
- Finally, the marginal models are conditional skewed-t distributions with certain GARCH-type conditional mean and variance on the information set $\mathbf{I}^{t-1}$.

Building the marginal models

Given the above-mentioned marginal models, we can obtained the PIT estimates by introducing the returns into the fitted conditional skewed t distributions, and implement the copula analysis by using these PITs.

We first check the null of independent copula. This is analogue to testing the null of serial independence as the first step in the time series analysis. Not surprisingly, this hypothesis is strongly rejected by the copula tests.

Static or dynamic copulae to use?

- We first check the null of CCC by using the test of Bera and Kim (2002, J. Empirical Finance).
- Then, we fit static (dynamic) copulae to the PIT combinations if the null of CCC is not rejected (is rejected).
- The dynamic copulae are assume to have the DCC of Tse and Tsui (2002) as discussed previously.
- We estimate the static and dynamic copulae by using the three-stage ML method; see Tables 10 and 11 for the MLEs.

Empirical findings

The M_τ test statistic becomes insignificant for all the combinations, regardless of whether C_N, C_G, C_G^s, or C_t is being tested.

Empirical findings

The M_τ test statistic becomes insignificant for all the combinations, regardless of whether C_N, C_G, C_G^s, or C_t is being tested.

This demonstrates that these bivariate copulae are all capable of interpreting the concordance structure of the return combinations captured by the positive and significant M_τ test statistics for the null of $C_o = C_I$.

The normal copula sharply outperforms the Gumbel and Gumbel-survival copulae.

- The M_{LU}, $M_L(u)$, and $M_U(u)$ tests accept the null of $C_o = C_N$ for sixteen out of the twenty return combinations; the only exceptions include SP**1**-FT**2**, SP**1**-CA**2**, RS**1**-TW**3**, and FT**1**-CA**1**.
- By contrast, the null of $C_o = C_G$ is accepted by all these tests for only three return combinations: SP**1**-NK**2**, SP**1**-TW**2**, and CA**1**-TW**5**.
- The null of $C_o = C_G^s$ is accepted by all these tests for seven return combinations: SP**1**-NK**2**, SP**1**-HS**2**, SP**1**-TW**2**, RS**2**-HS**3**, FT**1**-TW**4**, CA**1**-TW**5**, and NK**1**-TW**1**.

Moreover, the tail-dependence tests are unable to reject the null of $C_o = C_t$ for all the return combinations considered.

Moreover, the tail-dependence tests are unable to reject the null of $C_o = C_t$ for all the return combinations considered. Accordingly, we may characterize the cross-dependence structures of these return combinations (most of these combinations) by using the t copula (the normal copula).

Because C_N and C_t are both of the U-shaped dependence, this implies that the co-movements of these stock markets will be further strengthened in turbulent periods. Moreover, this structure should symmetrically hold for both the downside and upside markets, and hence does not support the hypothesis of "correlation asymmetry". We find a similar result in the trivariate copula analysis.

We find that the Gumbel (Gumbel-survival) copula tends to under-estimate (over-estimate) the lower tail-dependence but to over-estimate (under-estimate) the upper tail-dependence for large $|u|$'s in this empirical study. Interestingly, this test result is consistent with the dissimilarity between the J-shape (L-shape) dependence implied by the Gumbel (Gumbel-survival) copula being rejected and the U-shaped dependence implied by the normal (or t) copula being accepted.

This demonstrates that the $M_{L(u)}$ and $M_{U(u)}$ tests are useful in identifying the directions of copulae mis-specifications, as shown in the simulation.

Future Directions

- Testing: Dynamic misspecification.
- Specification: Maximum entropy + copula, High-dimensional copula.
- Out-of-sample: Forecasting + Cross-dependence.
- Applications: Macroeconomic time series, International business cycle, ...

Thank You.